How Likely A Coalition of Voters Can Influence A Large Election?

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Abstract

For centuries, it has been widely believed that the influence of a small coalition of voters is negligible in a large election. Consequently, there is a large body of literature on characterizing the asymptotic likelihood for an election to be influenced, especially by the *manipulation* of a single voter, establishing an $O(\frac{1}{\sqrt{n}})$ upper bound and an $\Omega(\frac{1}{n^{67}})$ lower bound for many commonly studied voting rules under the i.i.d. uniform distribution, known as *Impartial Culture (IC)* in social choice, where n is the number is voters.

In this paper, we extend previous studies in three aspects: (1) a more general and realistic *semi-random model*, (2) many *coalitional influence* problems, including coalitional manipulation, margin of victory, and various vote controls and bribery, and (3) arbitrary and variable coalition size B. Our main theorem provides asymptotically tight bounds on the semi-random likelihood of the existence of a size-B coalition that can successfully influence the election under a wide range of voting rules. Applications of the main theorem and its proof techniques resolve long-standing open questions about the likelihood of coalitional manipulability under IC, by showing that the likelihood is $\Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}\right)$ for many commonly studied voting rules.

Keywords

social choice; influence; semi-random models

1 Introduction

For centuries, it has been widely believed that the influence of a small coalition of voters is negligible in a large election. For example, Condorcet commented in 1785, that *"In single-stage elections, where there are a great many voters, each voter's influence is very small"* [12] (see the translation by McLean and Hewitt [45, pp.245]). As another example, Hegel commented in *The Philosophy of Right* in 1821, that *"the casting of a single vote is of no significance where there is a multitude of electors"* (see the translation and comments by Buchanan [10]).

Various types of influence were investigated in the literature. For example, *coalitional manipulation (*CM *for short)* refers to the phenomenon in which a coalition of voters have incentive to misreport their preferences, so that the winner is more favorable to all of them. The *margin of victory (*MoV *for short)* refers to the phenomenon in which a coalition of voters have the power to change the winner by voting differently, regardless of their incentives.

Accurately measuring the influence of voters is highly significant and plays a central role in many other studies. A small influence can be positive news or negative news, depending on the context. For example, a small influence is desirable under various robustness measures, such as decisiveness [29], strategy-proofness [28, 62], privacy [41]. On the other hand, a large influence is desirable in justifying the *voting power* of small groups of voters [5], and therefore would encourage voter turnout [19, 60].

Building a realistic model to accurately measure the influence of voters turns out to be a highly challenging mission. Clearly, a sensible and informative measure cannot be completely based on the worst-case analysis. Take coalitional manipulation with a single voter for example: for many voting rules, there always exists a situation where a voter can and has incentive to change the election outcome by changing his/her vote, due to the celebrated Gibbard-Satterthwaite theorem [28, 62]. Similarly, unless the voting rule always chooses the same alternative as the winner, there exists a situation where a single voter is highly influential, i.e., MoV = 1.

Consequently, there is a large body of literature on understanding the influence of voters using *average-case* analysis, especially for coaliational manipulation. Since the pioneering works by Pazner and Wesley [53] and Peleg [54] in the 1970's, much of the literature has focused on characterizing the asymptotic likelihood for randomly generated votes to be coalitional manipulable, as the number of voters $n \to \infty$. Previous work has established an $O(\frac{1}{\sqrt{n}})$ upper bound for many commonly-studied voting rules by a coalition of constantly many manipulators, under the i.i.d. uniform distribution over all linear orders, known as the *Impartial Culture (IC)* in social choice. On the other hand, the quantitative Gibbard-Satterthwaite theorems, e.g., [25, 47], established an $\Omega(\frac{1}{n^{67}})$ lower bound for a single manipulator under all voting rules that are constantly far away from dictatorships, under IC.

To the best of our knowledge, the only known matching lower bound is the $\Omega(\frac{1}{\sqrt{n}})$ bound for a single manipulator under the plurality rule [64]. Little is known for arbitrary size of the coalition, other means of coalitional influence such as margin of victory (MoV), vote controls, and bribery [21], and/or other statistical models for generating votes. Specifically, IC has been widely criticized of being unrealistic (see, e.g., [49, p. 30], [27, p. 104], and [36]), which means that the conclusion drawn under IC may only have limited implications in practice. See Section 1.2 for more discussions. Therefore, the following question remains largely open.

How likely a coalition of voters can influence large elections under realistic models?

The importance of answering this question has been widely recognized, as Pattanaik [52, p.187] discussed for (coalitional) manipulation soon after the discovery of the Gibbard-Satterthwaite theorem: *"For, if the likelihood of such strategic voting is negligible, then one need not be unduly worried about the existence of the possibility as such."*

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1.1 Our Contributions

We answer the question by characterizing the likelihood for many commonly studied voting rules and many commonly studied coalitional influence problems under a *semi-random* model [24] proposed in [76], which is inspired by and resembles the *smoothed analysis* [70] and is more general than IC.

The semi-random model in this paper. For any coalitional influence problem X studied in this paper, including coalitional manipulation (CM) and margin of victory (MoV), any voting rule r, any budget $B \geq 0$, and any profile P, we use a binary function $X(r, P, B)$ to indicate whether there exists a coalition of no more than B voters who can influence the outcome of r (with or without incentive) according to X. That is, $X(r, P, B) = 1$ if a group of no more than B voters are influential under P; otherwise $X(r, P, B) = 0$. For example, $\text{CM}(r, P, B) = 1$ if and only if there exist a coalition of no more than B voters who have incentive to misreport their preferences to improve the winner. Let Π denote a set of distributions over all rankings over the alternatives.

The *max-semi-random likelihood* of X under r with n agents and budget B, denoted by $\widetilde{X}^{\max}_{\Pi}(r, n, B)$, is defined as:

$$\widetilde{X}^{\max}_{\Pi}(r, n, B) \triangleq \sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}} X(r, P, B) \tag{1}$$

Similarly, the *min-semi-random likelihood* is defined as:

$$\widetilde{X}^{\min}_{\Pi}(r, n, B) \triangleq \inf_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}} X(r, P, B) \tag{2}$$

In other words, $\widetilde{X}^{\max}_{\Pi}(r, n, B)$ (respectively, $\widetilde{X}^{\min}_{\Pi}(r, n, B)$) upper-bounds (respectively, lower-bounds) the average-case likelihood of X, where the votes are generated independently (but not necessarily identically) from worst-case distribution vectors in Π^n. When a small influence is desirable (e.g., in the context of decisiveness, strategy-proofness, and privacy of voting), a low max-semi-random likelihood is good news, because it implies that the influence is small regardless of the underlying distributions $\vec{\pi}$. On the other hand, when a large influence is desirable (e.g., in the context of voting power and turnout), a large min-semi-random likelihood is good news.

The main result of this paper is the following characterization of semi-random coalitional influence under commonly studied voting rules.

Theorem 1. (Semi-random coalitional influence, informally put). *For many commonly studied influence problems X including* CM *and* MoV*, many commonly studied voting rules r with fixed number of alternatives $m \geq 3$, and any Π from a large class, there exists a constant $C_1 > 0$, such that for any $n \in \mathbb{N}$ and any $0 \leq B \leq C_1 n$, there exist non-negative integers $d^{\max}_{\Delta}$ and $d^{\max}_0$ that are no more than $m!$, such that*

$$\widetilde{X}^{\max}_{\Pi}(r, n, B) \text{ is } 0,\ \exp(-\Theta(n)),\text{ or } \Theta\left(\frac{(\min\{B+1, \sqrt{n}\})^{d^{\max}_{\Delta}}}{(\sqrt{n})^{m!-d^{\max}_0}}\right)$$

The full version of Theorem 1 also includes a similar characterization for $\widetilde{X}^{\min}_{\Pi}(r, n, B)$. The main merit of the theorem is conceptual, because it illustrates a separation between the 0 case, the exponential case, and the polynomial cases, for a large class of settings. The $B+1$ value in the numerator of the polynomial case is only used to handle the $B = 0$ case. For any $B \geq 1$, the numerator is effectively $\text{poly}(\min\{B, \sqrt{n}\})$.

Technically, the proof of Theorem 1 also characterizes the condition and $d^{\max}_{\Delta}$ and $d^{\max}_0$ for each case, but they are often not informative due to the generality of the theorem, and are therefore omitted in the theorem statement. Nevertheless, Theorem 1 and its proof techniques can be applied to accurately bound the semi-random likelihood in a wide range of settings of interest, sometimes resolving long-standing open questions. See Procedure 1 and discussions after it in Section 4 for the high-level idea. We present three examples of such applications in this paper.

The first application (Theorem 4) states that, for many commonly studied influence problems X including CM and MoV, many commonly studied voting rules r, such as any integer positional scoring rule, STV, ranked pairs, Schulze, maximin, or Copeland, a large class of Π including IC, any coalition size $B \geq 1$, and any sufficiently large n,

$$\widetilde{X}^{\max}_{\Pi}(r, n, B) = \Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}\right)$$

A straightforward application of Theorem 4 to CM under IC (Corollary 1) not only closes the $\left(\Omega(\frac{1}{n^{-67}}), O(\frac{1}{\sqrt{n}})\right)$ gap for these rules with $B = 1$ in previous work, but also provides asymptotically tight bounds on the likelihood of CM for every $B \geq 1$: roughly speaking, each additional manipulator (up to $O(\sqrt{n})$) increases the likelihood of success by $\Theta(\frac{1}{\sqrt{n}})$ under IC.

The second application (Theorem 5) establishes an $O\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}\right)$ upper bound on the likelihood of many types of coalitional influence, including CM and MoV, for all *generalized scoring rules (GSRs)* [79], which includes all voting rules mentioned in this paper. This supersedes all previous upper bounds discussed in this paper and extends them to arbitrary $B \geq 1$ and a more general (semi-random) model. Theorem 5 can be viewed as good news for CM, as it states that the likelihood vanishes for any $B = o(\sqrt{n})$ as $n \to \infty$. It also suggest that there is a large room in either designing natural voting rules with lower likelihood of CM, or improving the $\Omega(\frac{1}{n^{-67}})$ lower bound for voting rules that are constantly far away from dictatorships [47].

The third application (Theorem 6) investigates a new coalitional influence problem called *coalitional manipulation for the loser*, denoted by CML, which requires that a coalition of voters are incentivized to misreport their preferences to make the loser win. We prove that, for any integer positional scoring rules with $m \geq 3$ (except veto), any Π from a large class, and any sufficiently large n and B,

$$\widetilde{\text{CML}}^{\max}_{\Pi}(r_{\vec{s}}, n, B) = \Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}^{m-1}\right)$$

While CML may be of independent interest, the main purpose of this theorem is to illustrate that the likelihood of coalitional influence can be much smaller than $\Theta(\frac{B}{\sqrt{n}})$ or even $\Omega(\frac{1}{n^{-67}})$, can be nonlinear in B, and the degree of polynomial can depend on m. As suggested by Theorem 6, for CML, each additional manipulator is marginally more powerful when they are incentivized to make the loser win. We are not aware of a similar result in the literature.

The settings and (asymptotically tight) bounds of Theorem 1 and its three applications are summarized in Table 1.

	max/min	X	Π	Rule	B	Semi-random X
Thm. 1	both	CM, MoV, & etc. (Sec. B.2)	a large class	any GSR	$[0, O(n)]$	0, $\exp(-\Theta(n))$, or $\Theta\left(\frac{(\min\{B+1,\sqrt{n}\})^{d_\Delta}}{(\sqrt{n})^{m!-d_0}}\right)$
Thm. 4	max		$\pi_{\text{uni}} \in \text{CH}(\Pi)$	commonly studied GSRs	$[1, \infty)$	$\Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}\right)$
Thm. 5			a large class	any GSR	$[1, \infty)$	$O\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}\right)$
Thm. 6		CML	$\pi_{\text{uni}} \in \text{CH}(\Pi)$	Int. Pos. Sco. (except veto)	$[B^*, \infty)$	$\Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}^{m-1}\right)$

Table 1: Theorem 1 and its applications. X is a coalitional influence problem. In this paper, Π is assumed to be strictly positive and closed, see Section 2. "A large class" means that no additional assumption is made on Π. π_{uni} is the uniform distribution over all rankings. CH(Π) is the convex hull of Π. GSR represents *generalized scoring rules*, see Section 2. B^* is a constant that does not depend on B or n. d_0 and d_Δ are non-negative integers that are no more than $m!$.

Techniques. Notice that the histogram of randomly generated votes, called a preference profile, is a *Poisson multivariate variable (PMV)*, formally defined as follows.

Definition 1 **(Poisson multivariate variable (PMV))**. *Given $n, q \in \mathbb{N}$ and a vector $\vec{\pi} = (\pi_1, \ldots, \pi_n)$ of n distributions over $\{1, \ldots, q\}$, an (n, q)-PMV is denoted by $\vec{X}_{\vec{\pi}}$, which represents the histogram of n independent random variables whose distributions are $\{\pi_1, \ldots, \pi_n\}$, respectively.*

It turns out that many coalitional influence problems under commonly-studied voting rules can be modeled as *PMV-instability problems* (Definition 4), where we are given a *source* polyhedron $\mathcal{H}_S$, a *target* polyhedron $\mathcal{H}_T$, a set of *vote operations* $\mathbb{O} \subseteq \mathbb{R}^q$, each of which represents a type of changes to the histogram by the influencers, and a cost vector $\vec{c}$ for the vote operations in $\mathbb{O}$. Then, given n and B, we are interested in the semi-random likelihood for a PMV to be *unstable*, in the sense that it is in $\mathcal{H}_S$ and can be changed to $\mathcal{H}_T$ by performing vote operations in $\mathbb{O}$ under a budget constraint B. We prove the following characterization of the PMV-instability problem.

Theorem 2. (Semi-random instability of PMV, informally put). *For any PMV-instability problem, a large class of Π, any $n \in \mathbb{N}$, and any $B \geq 0$, the upper bound (respectively, lower bound) on the PMV-instability problem falls into one of the following four cases:*

$0, \exp(-\Theta(n))$, *phase transition at* $\Theta(\sqrt{n})$, *and phase transition at* $\Theta(n)$

In the phase-transition-at-$\Theta(\sqrt{n})$ case, the likelihood increases to its maximum $\text{poly}^{-1}(n)$ before $B = \Theta(\sqrt{n})$. In the phase-transition-at-$\Theta(n)$ case, the likelihood increases from $\exp(-\Theta(n))$ to its maximum $\text{poly}^{-1}(n)$ around $B = \Theta(n)$. The formal statement of Theorem 2 also characterizes conditions and asymptotically tight bounds for the four cases.

Then, Theorem 1 naturally follows after Theorem 2 (more precisely, its extension Theorem 3 to the union of finitely many PMV-instability problems).

1.2 Related Work and Discussions

Due to the large body of literature, below we briefly discuss some closely-related work. More detailed discussions can be found in Appendix A.

Likelihood of coalitional manipulability: upper bounds w.r.t. IC. As discussed earlier, a series of work has established an $O(\frac{1}{\sqrt{n}})$ upper bound on the likelihood of single-voter manipulability under commonly studied voting rules w.r.t. IC [4, 26, 35, 43, 48, 51, 53, 54, 57, 64–67]. The likelihood of coalitional manipulability was also considered for a few rules with certain sizes of the coalition [53, 54, 57, 67]. All these results are straightforward corollaries of our Theorem 5, which implies an $O\left(\min\left(\frac{B}{\sqrt{n}}, 1\right)\right)$ upper bound for every $B \geq 1$ w.r.t. IC under a wide range of voting rules, including (but not limited to) the ones studied in previous work.

Likelihood of coalitional manipulability: lower bounds w.r.t. IC. A series of work on quantitative Gibbard-Satterthwaite theorems [18, 25, 33, 47, 80] established polynomial lower bounds on the likelihood of strategic voting by a single manipulator ($B = 1$) w.r.t. IC. Many of these results applied to any voting rule, and are therefore more general than the CM part of Theorem 4 and Corollary 1. On the other hand, Theorem 4 is stronger and more general in some other aspects: it works for every $B \geq 1$, more coalitional influence problems, and a more general and realistic (semi-random) model, and it provides asymptotically tight bounds. Its application to CM w.r.t. IC in Corollary 1 establishes an $\Omega(\frac{1}{\sqrt{n}})$ matching bound for many commonly studied rules, among which this matching lower bound was only known for the plurality rule [64].

Likelihood of coalitional manipulability: other distributions. Results under i.i.d. distributions [46, 54, 58, 75, 79] demonstrated a phase transition from powerless to powerful at $\Theta(\sqrt{n})$ manipulators. There is a large body of literature on the likelihood of CM under the *Impartial Anonymous Culture (IAC)*, which assumes that each histogram happens equally likely and resembles the flat Dirichlet distribution, by theoretical analysis [22, 23, 32, 37–39, 61, 68, 74] and by computer simulations [31, 40, 56]. Both IAC and IC are mainly of theoretical interest and *"are poor proxies of political electorates"* [50].

Other coalitional influence problems. All results in this paper also work for MoV, which measures the stability of elections and provides an upper bound on CM. Some previous proofs of upper bounds on CM were done for MoV [64, 79]. The phase transition behavior for CM also happens for MoV [46, 75]. Previous work has also investigated expected MoV [55] and MoV for tournament rules [9]. The likelihood of MoV = 1 has been used to measure the *decisiveness* of voting, sometimes called *voting power*, which plays an important role in the paradox of voting [19] and in power indices in cooperative game theory. For two alternatives under the plurality rule, the voting power is equivalent to the likelihood of ties [30]. In general, MoV = 1 and the election being tied are different events, as shown in Example 6 and 7 in Appendix H.

Beyond CM and MoV, there is a large body of work on the computational complexity of other types of coalition influence problems, such as constructive/destructive control (making a designated alternative win/lose) by adding/deleting votes, and bribery (different changes in votes have different costs). See [21] for a recent survey and see Appendix B.2 for their definitions. Little was known about their likelihood of success, except [75], which does not provide an accurate characterization as discussed in Appendix A. The full versions of our results in the Appendix provide the first asymptotically tight bounds for these coalitional influence problems.

Semi-random analysis and smoothed analysis. Semi-random analysis [7, 8] refers to the analysis under a model where the process of generating instances has adversarial components and random components. For example, in the smoothed complexity analysis [70], the input to an algorithm is obtained from an adversarially chosen "ground truth" plus a (small) random perturbation. This can be viewed as the adversary directly choosing a distribution over data (from a set of distributions). See [24] for a recent survey of various semi-random models and complexity results under them. In this paper, we adopt the semi-random model proposed in [76]. See Section 2 for more discussions on its generality and limitations.

Technical novelty. While this paper takes a similar "polyhedral" approach adopted in previous work on semi-random social choice [76–78], the main technical tool of this paper (Theorem 2) is a significant and non-trivial extension of the main technical theorems in previous work, especially [77, Theorem 1], which can be viewed as a special case of our Theorem 2 with $\mathcal{H}_S = \mathcal{H}_T$ and $B = 0$. The hardest part is the polynomial lower bounds, whose $\mathcal{H}_S = \mathcal{H}_T$ and $B = 0$ case was proved in [77, Theorem 1] by explicitly enumerating sufficiently many target integer vectors for the PMV. However, due to the generality of the PMV-instability problem, we do not see an easy way to perform a similar enumeration. To address this technical challenge, we take a different approach by first pretending that the PMV can take non-integer values, then enumerating (possibly non-integral) vectors that are far away from each other by exploring two directions: the direction that represents no budget (i.e., $B = 0$) and the direction that represents infinite budget (i.e., $B = \infty$), and finally proving that for each such (possibly non-integral) vector, there exists a nearby integer target vector for the PMV. See Section C.3 for the intuitions and a proof sketch, and Appendix E.2 for the full proof. We believe that our Theorem 2 is a useful and general tool for studying likelihood of coalitional influence, as exemplified by its applications to prove Theorem 1, Theorems 4–6, and Corollary 1.

2 Preliminaries

For any $q \in \mathbb{N}$, let $[q] = \{1, \ldots, q\}$. Let $\mathcal{A} = [m]$ denote a set of $m \geq 3$ *alternatives*. Let $\mathcal{L}(\mathcal{A})$ denote the set of all linear orders over $\mathcal{A}$. Let $n \in \mathbb{N}$ denote the number of agents (voters). Each agent uses a linear order $R \in \mathcal{L}(\mathcal{A})$ to represent his or her preferences, called a *vote*, where $a \succ_R b$ or $\{a\} \succ_R \{b\}$ means that the agent prefers alternative a to alternative b. The vector of n agents' votes, denoted by P, is called a *(preference) profile*, sometimes called an n-profile. The set of n-profiles for all $n \in \mathbb{N}$ is denoted by $\mathcal{L}(\mathcal{A})^* = \bigcup_{n=1}^{\infty} \mathcal{L}(\mathcal{A})^n$. A *fractional* profile is a profile P coupled with a possibly non-integer and/or negative weight vector $\vec{\omega}_P = (\omega_R : R \in P) \in \mathbb{R}^n$ for the votes in P. Sometimes the weight vector is omitted when it is clear from the context.

For any (fractional) profile P, let $\text{Hist}(P) \in \mathbb{R}_{\geq 0}^{m!}$ denote the anonymized profile of P, also called the *histogram* of P, which contains the total weight of every linear order in $\mathcal{L}(\mathcal{A})$ according to P. An *irresolute voting rule* $\overline{r} : \mathcal{L}(\mathcal{A})^* \to (2^{\mathcal{A}} \setminus \{\emptyset\})$ maps a profile to a non-empty set of winners in $\mathcal{A}$. A *resolute* voting rule r is a special irresolute voting rule that always chooses a single alternative as the (unique) winner. Often a resolute rule is obtained from an irresolute rule by applying a *tie-breaking mechanism*, e.g., lexicographic tie-breaking, which chooses the co-winner with the smallest index as the unique winner.

Integer positional scoring rules. An *(integer) positional scoring rule* $\overline{r}_{\vec{s}}$ is characterized by an integer scoring vector $\vec{s} = (s_1, \ldots, s_m) \in \mathbb{Z}^m$ with $s_1 \geq s_2 \geq \cdots \geq s_m$ and $s_1 > s_m$. For any alternative a and any linear order $R \in \mathcal{L}(\mathcal{A})$, we let $\vec{s}(R, a) = s_i$, where i is the rank of a in R. Commonly studied integer positional scoring rules include *plurality*, which uses the scoring vector $(1, 0, \ldots, 0)$, *Borda*, which uses the scoring vector $(m-1, m-2, \ldots, 0)$, and *veto*, which uses the scoring vector $(1, \ldots, 1, 0)$.

Generalized scoring rules (GSRs). All voting rules studied in this paper are *generalized scoring rules (GSRs)* [79]. We recall the definition of GSRs based on separating hyperplanes [46, 81] as follows.

Definition 2. *A* generalized scoring rule (GSR) r *is defined by (1) a set of $K \geq 1$ hyperplanes $\vec{H} = (\vec{h}_1, \ldots, \vec{h}_K) \in (\mathbb{R}^{m!})^K$ and (2) a function $g : \{+, -, 0\}^K \to \mathcal{A}$. For any profile P, we let $r(P) = g(\text{Sign}_{\vec{H}}(\text{Hist}(P)))$, where $\text{Sign}_{\vec{H}}(\vec{x}) = (\text{Sign}(\vec{h}_1 \cdot \vec{x}), \ldots, \text{Sign}(\vec{h}_K \cdot \vec{x}))$ represents the signs of $\vec{h}_1 \cdot \vec{x}, \ldots, \vec{h}_K \cdot \vec{x}$. When $\vec{H} \in (\mathbb{Z}^{m!})^K$, r is called an* integer GSR (int-GSR).

Example 1 (**Borda as a GSR**). *Let $m = 3$. Borda with lexicographic tie-breaking is a GSR with $K = m$ and $\vec{H} = \{\vec{h}_1, \vec{h}_2, \vec{h}_3\}$ defined as follows.*

$$\begin{array}{rcrrrrrrc} & & x_{123} & x_{132} & x_{213} & x_{231} & x_{312} & x_{321} & \\ \vec{h}_1 = & (& 1, & 2, & -1, & -2, & 1, & -1 &) \\ \vec{h}_2 = & (& 2, & 1, & 1, & -1, & -1, & -2 &) \\ \vec{h}_3 = & (& 1, & -1, & 2, & 1, & -2, & -1 &) \end{array}$$

Let $\vec{x} = (x_{123}, x_{132}, x_{213}, x_{231}, x_{312}, x_{321})$ denote the histogram of a profile, where x_{123} represents the number of $[1 \succ 2 \succ 3]$ votes. It follows that $\vec{h}_1 \cdot \vec{x}$ is the Borda score of alternative 1 minus the Borda score of 2 in the profile; $\vec{h}_2 \cdot \vec{x}$ is the Borda score of alternative 1 minus the Borda score of 3 in the profile; and $\vec{h}_3 \cdot \vec{x}$ is the Borda score of alternative 2 minus the Borda score of 3 in the profile. The g function chooses the winner based on $\text{Sign}_{\vec{H}}(\text{Hist}(P))$ and the tie-breaking mechanism.

Coalitional influence problems. Let r be a resolute rule, P be a preference profile, and $B \geq 0$ be a budget. *Coalitional manipulation (*CM*)* is defined by a binary function $\text{CM}(r, P, B)$ such that $\text{CM}(r, P, B) = 1$ if and only there exists $P' \subseteq P$ with $|P'| \leq B$ and P^* with $|P^*| = |P'|$, such that for all $R \in P'$, $r(P - P' + P^*) \succ_R r(P)$. The *margin of victory (*MoV*)* is a function $\text{MoV}(r, P, B)$ such that $\text{MoV}(r, P, B) = 1$ if and only if no more than B voters can coalitionally change the winner (regardless of their preferences and incentives).

See Appendix B.1 for definitions of some other commonly studied voting rules (which are GSRs), i.e., ranked pairs, Schulze, maximin, Copeland, and STV, and Appendix B.2 for the definitions of some other commonly-studied coalitional manipulation problems, i.e., constructive/destructive control by adding/deleting votes and bribery. Many results in this paper apply to these rules and coalitional influence problems, as stated in their full versions in the Appendix.

Semi-random likelihood of coalitional influence. As discussed in the Introduction, given a coalitional influence problem X, a set Π of distributions over $\mathcal{L}(\mathcal{A})$, a voting rule r, a number of voters $n \in \mathbb{N}$, and a budget constraint $B \geq 0$, the *max-semi-random likelihood of* X (*max-semi-random* X for short), denoted by $\widetilde{X}^{\max}_{\Pi}(r, n, B)$, is defined in Equation (1). Similarly, the *min-semi-random likelihood of* X (*min-semi-random* X for short), denoted by $\widetilde{X}^{\min}_{\Pi}(r, n, B)$, is defined in Equation (2).

Assumptions on Π. Throughout the paper, we make the following two assumptions on Π: (1) *strict positiveness*, which means that there exists a constant $\epsilon > 0$ such that the probabilities over all rankings in all $\pi \in \Pi$ are larger than ϵ, and (2) *closedness*, which means that Π is a closed set in the Euclidean space.

Clearly, IC corresponds to $\Pi = \{\pi_{\text{uni}}\}$, where π_{uni} is the uniform distribution over $\mathcal{L}(\mathcal{A})$. Let us take a look at another example of Π and its corresponding semi-random likelihood.

Example 2 (**Semi-random** CM **under Borda**). *Let X = CM and r = Borda with lexicographic tie-breaking. Let $\mathcal{A} = \{1, 2, 3\}$ and $\Pi = \{\pi^1, \pi^2\}$, where π^1 and π^2 are distributions in Table 2.*

	123	132	231	321	213	312
π^1	1/4	1/4	1/4	1/12	1/12	1/12
π^2	1/12	1/12	1/12	1/4	1/4	1/4

Table 2: Π in Example 2.

When $n = 2$ and $B = 1$, we have

$$\widetilde{\text{CM}}^{\max}_{\Pi}(\textit{Borda}, 2, 1) = \sup_{\vec{\pi} \in \{\pi^1, \pi^2\}^n} \Pr_{P \sim \vec{\pi}} \text{CM}(\textit{Borda}, P)$$

That is, the adversary has four choices of $\vec{\pi}$, i.e., $\{(\pi^1, \pi^1), (\pi^1, \pi^2), (\pi^2, \pi^1), (\pi^2, \pi^2)\}$. Each $\vec{\pi}$ leads to a distribution over the set of all profiles of two agents, i.e., $\mathcal{L}(\mathcal{A})^2$. As we will see later in Example 5, for every sufficiently large n, $\widetilde{\text{CM}}^{\max}_{\Pi}(\textit{Borda}, n, 1) = \Theta(\frac{1}{\sqrt{n}})$.

Generality and limitations of the semi-random model. There are two major limitations of the semi-random model studied in this paper: the independence of noises among agents' preferences, and the strict positiveness of Π. While results on general models are always desirable, these limitations may not be as strong as they appear in the social choice context, due to the following reasons.

• First, the independence of agents' noises is a common assumption in classical models for human's preferences and behaviors, such as *random utility models* [71] and *discrete choice models* [72]. Notice that in our semi-random model, the adversary is allowed to choose any combination of distributions in Π for the agents, which means that the agents' "de-noised" preferences can be arbitrarily correlated.

• Second, straightforward relaxations of strict positiveness of Π easily leads to trivial and negative results. For example, if we allow some probabilities to be 0, then Π may contain "deterministic" distributions that have 100% probability on some rankings, and in such cases the semi-random analysis degenerates to the worst-case analysis.

After all, we believe that the semi-random model studied in this paper, which extends IC, is a step forward towards a more realistic measure of voters' influence. How to incorporate dependent noises among agents and how to relax the strict positiveness of Π (e.g., by allowing the lower bound on probabilities in Π to depend on n, such as $\frac{1}{n}$) are important and challenging directions for future research.

3 Main Result: Semi-Random Coalitional Influence

Theorem 1. *Let r denote any int-GSR with fixed $m \geq 3$. For any closed and strictly positive Π and any $X \in \{\text{CM}, \text{MoV}\}$, there exists a constant $C_1 > 0$, such that for any $n \in \mathbb{N}$ and any $B \geq 0$ with $B \leq C_1 n$, there exist $\{d_0^{\max}, d_\Delta^{\max}, d_0^{\min}, d_\Delta^{\min}\} \subseteq [m!]$ such that*

$$\widetilde{X}^{\max}_{\Pi}(r, n, B) \text{ is } 0,\ \exp(-\Theta(n)),\ \text{or } \Theta\left(\frac{\min\{B+1, \sqrt{n}\}^{d_\Delta^{\max}}}{(\sqrt{n})^{m! - d_0^{\max}}}\right), \text{ and}$$
$$\widetilde{X}^{\min}_{\Pi}(r, n, B) \text{ is } 0,\ \exp(-\Theta(n)),\ \text{or } \Theta\left(\frac{\min\{B+1, \sqrt{n}\}^{d_\Delta^{\min}}}{(\sqrt{n})^{m! - d_0^{\min}}}\right)$$

As explained in the Introduction, the main merit of Theorem 1 is conceptual, as it illustrates a separation between 0, exponential, and polynomial cases of different degrees. The most interesting part is the asymptotically tight polynomial lower bounds, because little was known about them even under CM, IC, and $B = 1$, as discussed in Section 1.2. The theorem works for many other commonly-studied coalitional influence problem such as those defined in Appendix B.2, as indicated in the full version of the theorem in Appendix C.2.

Overview of the proof of Theorem 1. The proof proceeds in three steps. In Step 1 (Section 3.1), we model commonly-studied coalitional influence problems, including CM and MoV, under GSRs as unions of multiple *PMV-instability problems* (and the union is formally defined as *PMV-multi-instability problems* in Definition 6). Then in Step 2 (Section C.3), we characterize the semi-random likelihood of the PMV-instability problem in Theorem 2. In Step 3, we first extend Theorem 2 to solve the PMV-multi-instability problem in Theorem 3, and then apply it to the PMV-multi-instability problems defined in Step 1 to prove Theorem 1. The full proof can be found in Appendix C.2.

$$\left.\begin{array}{r}-x_{123}-2x_{132}+x_{213}+2x_{231}-x_{312}+x_{321}\leq\ 0\\ -2x_{123}-x_{132}-x_{213}+x_{231}+x_{312}+2x_{321}\leq\ 0\\ -\vec{x}\leq\ 0\end{array}\right\}\text{1 wins before manipulation}$$

$$\left.\begin{array}{r}x_{123}+2x_{132}-(x_{213}-o_1)-2(x_{231}+o_1+o_2)+x_{312}-(x_{321}-o_2)\leq -1\\ -x_{123}+x_{132}-2(x_{213}-o_1)-(x_{231}+o_1+o_2)+2x_{312}+(x_{321}-o_2)\leq\ 0\\ -o_1\leq 0,-o_2\leq 0,-(x_{213}-o_1)\leq 0,-(x_{321}-o_2)\leq 0,o_1+o_2\leq\ B\end{array}\right\}\text{2 wins after manipulation}$$

Figure 1: The linear system for $\mathcal{U}_{n,B}^{1\to 2}$ in Example 3.

3.1 Step 1: Modeling

We start with an example of modeling CM as systems of linear inequalities, which motivates the study of the more general *PMV-instability problem* (Definition 4) and its extension *PMV-multi-instability problem* (Definition 6).

Example 3. *Let $\mathcal{A}=\{1,2,3\}$ and let r be Borda with lexicographic tie-breaking. Let $\mathcal{U}_{n,B}^{1\to 2}$ denote the histograms of all n-profiles that satisfy the following conditions: (1) the winner before manipulation is 1, (2) a coalition of no more than B manipulators are motivated to change the winner to 2 by casting different votes. Notice that only the $[2\succ 1\succ 3]$ voters and the $[3\succ 2\succ 1]$ voters have incentive to misreport their preferences (both to $[2\succ 3\succ 1]$). Let o_1 (respectively, o_2) denote the number of voters who change their votes from $[2\succ 1\succ 3]$ (respectively, $[3\succ 2\succ 1]$) to $[2\succ 3\succ 1]$.*

Then, the histogram $\vec{x}$ of an n-profiles is in $\mathcal{U}_{n,B}^{1\to 2}$ if and only if there exists an integer vector $\vec{o}=(o_1,o_2)$ such that $(\vec{x},\vec{o})$ is a feasible solution to the linear program illustrated in Figure 1, where the objective is omitted because only the feasibility matters.

In Example 3, the effect of each manipulator can be modeled by its changes to the histogram, and the manipulators aim at manipulating vectors in a *source* polyhedron, which represents 1 being the winner, into a *target* polyhedron, which represents 2 being the winner, under the budget constraint B. This motivates us to define the *PMV-instability setting* as follows.

Definition 3 (**PMV-instability setting**). *In a* PMV-instability setting $\mathcal{S}\triangleq\langle\mathcal{H}_S,\mathcal{H}_T,\mathbb{O},\vec{c}\,\rangle$,

- *$\mathcal{H}_S$ and $\mathcal{H}_T$ are polyhedra in $\mathbb{R}^q$ for some $q\in\mathbb{N}$, the subscript S and T represent "source" and "target", respectively. For $Y\in\{S,T\}$, let $\mathcal{H}_Y\triangleq\left\{\vec{x}\in\mathbb{R}^q:\mathbf{A}_Y\times(\vec{x})^\top\leq(\mathbf{b}_Y)^\top\right\}$, where $\mathbf{A}_Y$ is an integer matrix of q columns;*
- *$\mathbb{O}\subseteq\mathbb{R}^q$ is a finite set of* vote operations *[75], and let* $\mathbf{O}$ *denote the $|\mathbb{O}|\times q$ matrix whose rows are the vectors in $\mathbb{O}$;*
- *$\vec{c}\in\mathbb{R}_{\geq 0}^{|\mathbb{O}|}$ is the cost vector for the vote operations in $\mathbb{O}$.*

W.l.o.g., in this paper we assume $\vec{c}>\vec{0}$ and the minimum cost of a single operation is 1, i.e., $\min_{i\leq q}[\vec{c}\,]_i=1$. Given a PMV-instability setting $\mathcal{S}$, $n\in\mathbb{N}$, and a budget $B\geq 0$, we let $\mathcal{U}_{n,B}$ denote the set of non-negative size-n integer vectors that represent *unstable* histograms w.r.t. vote operations in $\mathbb{O}$ and budget B. That is,

$$\mathcal{U}_{n,B}\triangleq\left\{\underbrace{\vec{x}\in\mathcal{H}_S\cap\mathbb{Z}_{\geq 0}^q}_{\vec{x}\text{ is in }\mathcal{H}_S}:\underbrace{\vec{x}\cdot\vec{1}=n}_{n\text{-profile}}\text{ and }\exists\ \underbrace{\vec{o}\in\mathbb{Z}_{\geq 0}^{|\mathbb{O}|}}_{\text{vote operations}}\text{ s.t. }\underbrace{\vec{c}\cdot\vec{o}\leq B}_{\text{budget constraint}}\text{ and }\underbrace{\vec{x}+\vec{o}\times\mathbf{O}\in\mathcal{H}_T}_{\text{manipulated to be in }\mathcal{H}_T}\right\}$$

Example 4. *In the setting of Example 3, $q=m!=6$. $\mathcal{U}_{n,B}^{1\to 2}$ is the set of unstable histograms of the PMV-instability setting where $\mathcal{H}_S$ (respective, $\mathcal{H}_T$) is the polyhedron that represents 1 (respectively, 2) being the winner, $\mathbb{O}=\{(0,0,-1,1,0,0),(0,0,0,1,0,-1)\}$ (the indices to rankings are the same as in Example 1), $\mathbf{O}=\begin{bmatrix}0,&0,&-1,&1,&0,&0\\0,&0,&0,&1,&0,&-1\end{bmatrix}$, $\vec{c}=(1,1)$.*

We are interested in solving the *PMV-instability problem* defined as follows.

Definition 4 (**PMV-instability problem**). *Given a PMV-instability setting $\mathcal{S}=\langle\mathcal{H}_S,\mathcal{H}_T,\mathbb{O},\vec{c}\,\rangle$, a set Π of distributions over $[q]$, $n\in\mathbb{N}$, and $B\geq 0$, we are asked to bound*

max-semi-random instability: $\sup_{\vec{\pi}\in\Pi^n}\Pr\left(\vec{X}_{\vec{\pi}}\in\mathcal{U}_{n,B}\right)$, *and*

min-semi-random instability: $\inf_{\vec{\pi}\in\Pi^n}\Pr\left(\vec{X}_{\vec{\pi}}\in\mathcal{U}_{n,B}\right)$

That is, the max-semi-random instability (respectively, min-semi-random instability) is the upper bound (respectively, lower bound) on the probability for the PMV to be unstable, when the underlying probabilities $\vec{\pi}$ are adversarially chosen from Π^n.

Notice that Example 3 only captures the coalitional manipulation situations where alternative 1 is manipulated to alternative 2. Similarly, we can define $\mathcal{U}_{n,B}^{1\to 3}$, $\mathcal{U}_{n,B}^{2\to 1}$, $\mathcal{U}_{n,B}^{2\to 3}$, $\mathcal{U}_{n,B}^{3\to 1}$, and $\mathcal{U}_{n,B}^{3\to 2}$. Let $\mathcal{M}$ denote the set of the six PMV-instability settings, and let

$$\mathcal{U}_{n,B}^{\mathcal{M}}=\mathcal{U}_{n,B}^{1\to 2}\cup\mathcal{U}_{n,B}^{1\to 3}\cup\mathcal{U}_{n,B}^{2\to 1}\cup\mathcal{U}_{n,B}^{2\to 3}\cup\mathcal{U}_{n,B}^{3\to 1}\cup\mathcal{U}_{n,B}^{3\to 2}$$

Then, we have

$$\widetilde{\text{CM}}_\Pi^{\max}(\text{Borda},n,B)=\sup_{\vec{\pi}\in\Pi^n}\Pr\left(\vec{X}_{\vec{\pi}}\in\mathcal{U}_{n,B}^{\mathcal{M}}\right),\text{ and}$$

$$\widetilde{\text{CM}}_\Pi^{\min}(\text{Borda},n,B)=\inf_{\vec{\pi}\in\Pi^n}\Pr\left(\vec{X}_{\vec{\pi}}\in\mathcal{U}_{n,B}^{\mathcal{M}}\right)$$

This motivates us to define the *PMV-multi-instability setting*.

Definition 5 (**PMV-multi-instability setting**). *A* PMV-multi-instability setting*, denoted by $\mathcal{M}=\{\mathcal{S}^i:i\leq I\}$, is a set of $I\in\mathbb{N}$*

PMV-instability settings, where $\mathcal{S}^i = \langle \mathcal{H}_S^i, \mathcal{H}_T^i, \mathbb{O}^i, \vec{c}^{\,i}\rangle$, whose unstable histograms are denoted by $\mathcal{U}_{n,B}^i$. Let $\mathcal{U}_{n,B}^{\mathcal{M}} = \bigcup_{i\le I}\mathcal{U}_{n,B}^i$.

It turns out that many coalitional influence problems, such as CM under Borda discussed above, can be modeled by PMV-multi-instability settings, as shown in the following lemma.

Lemma 1 (**Coalitional Influence as PMV-multi-instability**). *For any $X \in \{\mathrm{CM}, \mathrm{MoV}\}$ and any GSR r, there exist a PMV multi-instability setting $\mathcal{M} = \{\mathcal{S}^i : i \le I\}$ such that for every n-profile P and every $B \ge 0$, $X(r,P,B) = 1$ if and only if $Hist(P) \in \mathcal{U}_{n,B}^{\mathcal{M}}$.*

The full statement of the lemma (which covers other coalitional influence problems) and its proof can be found in Appendix D.1. In light of Lemma 1, the (max- or min-) semi-random likelihood of commonly studied coalitional influence problems can be reduced to the following problem.

Definition 6 (**The PMV-multi-instability problem**). *Given a PMV-multi-instability setting $\mathcal{M} = \{\mathcal{S}^i : i \le I\}$, a set Π of distributions over $[q]$, $n \in \mathbb{N}$, and $B \ge 0$, we are asked to bound*

max-semi-random multi-instability: $\sup_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}^{\mathcal{M}}\right)$

min-semi-random multi-instability: $\inf_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}^{\mathcal{M}}\right)$

Like PMV-instability problems, the max-(respectively, min-) semi-random multi-instability represents the upper bound (respectively, lower bound) on the likelihood for the PMV to be unstable w.r.t. any $\mathcal{S}^i$ in $\mathcal{M}$, when the underlying probabilities $\vec{\pi}$ is adversarially chosen from Π^n.

3.2 Sketch of Step 2

To present the theorem, we introduce some notation, whose intuition is presented in Appendix C.3.

Notation. For any budget $B \ge 0$, we let $\mathcal{H}_B$ denote the relaxation of $\mathcal{U}_{n,B}$ by removing the size constraint and the integrality constraints on $\vec{x}$ and on $\vec{o}$. Recall that $\vec{o}$ still needs to be non-negative. That is,

$$\mathcal{H}_B \triangleq \left\{\vec{x} \in \mathcal{H}_{\mathrm{S}} : \exists \vec{o} \in \mathbb{R}_{\ge 0}^{|\mathbb{O}|} \text{ s.t. } \vec{c}\cdot\vec{o} \le B \text{ and } \vec{x} + \vec{o}\times \mathbf{O} \in \mathcal{H}_{\mathrm{T}}\right\}$$

For example, Figure 2 (a) illustrates $\mathcal{H}_B$ in the shaded area. $\mathcal{H}_B$ is a polyhedron because it is the intersection of $\mathcal{H}_{\mathrm{S}}$ and the Minkowski addition of $\mathcal{H}_{\mathrm{T}}$ and the following polyhedron Q_B:

$$Q_B \triangleq \left\{-\vec{o}\times\mathbf{O} : \vec{o} \in \mathbb{R}_{\ge 0}^{|\mathbb{O}|} \text{ and } \vec{c}\cdot\vec{o} \le B\right\}$$

Specifically, with infinite budget ($B = \infty$), we have

$$\mathcal{H}_\infty = \left\{\vec{x} \in \mathcal{H}_{\mathrm{S}} : \exists \vec{o} \in \mathbb{R}_{\ge 0}^{|\mathbb{O}|} \text{ s.t. } \vec{x} + \vec{o}\times\mathbf{O} \in \mathcal{H}_{\mathrm{T}}\right\} = \mathcal{H}_{\mathrm{S}} \cap (\mathcal{H}_{\mathrm{T}} + Q_\infty)$$

For every $B \ge 0$, we define C_B to be the polyhedron that consists of all (possibly non-integer) vectors in $\mathcal{H}_{\mathrm{S},\le 0}$ that can be manipulated to be in $\mathcal{H}_{\mathrm{T},\le 0}$ by using (possibly non-integer) operations $\vec{o}$ under budget constraint B. That is,

$$C_B \triangleq \left\{\vec{x} \in \mathcal{H}_{\mathrm{S},\le 0} : \exists \vec{o} \in \mathbb{R}_{\ge 0}^{|\mathbb{O}|} \text{ s.t. } \vec{c}\cdot\vec{o} \le B \text{ and } \vec{x} + \vec{o}\times\mathbf{O} \in \mathcal{H}_{\mathrm{T},\le 0}\right\}$$

It is not hard to verify that $C_B = \mathcal{H}_{\mathrm{S},\le 0} \cap (\mathcal{H}_{\mathrm{T},\le 0} + Q_B)$ and C_B can be viewed as a "pseudo-conic" approximation to $\mathcal{H}_B$, as C_B is defined based on the characteristic cones of $\mathcal{H}_{\mathrm{S}}$ and $\mathcal{H}_{\mathrm{T}}$, though C_B itself may not be a cone. Specifically, C_0 and C_∞ will play a central role in Theorems 2 and 3. It is not hard to verify that

$$C_0 \triangleq \mathcal{H}_{\mathrm{S},\le 0} \cap \mathcal{H}_{\mathrm{T},\le 0}, \text{ and}$$

$$C_\infty \triangleq \left\{\vec{x} \in \mathcal{H}_{\mathrm{S},\le 0} : \exists \vec{o} \in \mathbb{R}_{\ge 0}^{|\mathbb{O}|} \text{ s.t. } \vec{x} + \vec{o}\times\mathbf{O} \in \mathcal{H}_{\mathrm{T},\le 0}\right\}$$

Figure 2 (b) illustrates C_0 (which is a line) and C_∞ (which is the same as $\mathcal{H}_{\mathrm{S},\le 0}$). Both C_0 and C_∞ are polyhedral cones, because the intersection Minkowski addition of two polyhedral cones is a polyhedral cone. Notice that when $B \notin \{0,\infty\}$, C_B may not be a cone.

For any set $\Pi^* \subseteq \mathbb{R}^q$, let $B_{\Pi^*} \in \mathbb{R}$ to be the minimum budget B such that the intersection of Π^* and C_B is non-empty. If no such B exists (i.e., $\Pi^* \cap C_\infty = \emptyset$), then we let $B_{\Pi^*} \triangleq \infty$. Formally,

$$B_{\Pi^*} \triangleq \inf\{B^* \ge 0 : \Pi^* \cap C_B \ne \emptyset\}$$

Figure 2 (b) illustrates $B_{\mathrm{CH}(\Pi)}$ and $C_{B_{\mathrm{CH}(\Pi)}}$ in the shaded area, where $\Pi = \{\pi^1, \pi^2\}$ and $\mathrm{CH}(\Pi)$ is the *convex hull* of Π, which is the line segment between π^1 and π^2 in this case.

For any set $\Pi^* \subseteq \mathbb{R}^q$, we define $B_{\Pi^*}^- \in \mathbb{R}$ to be the minimum budget B such that Π^* is completely contained in C_B. If no such B exists, then we let $B_{\Pi^*}^- \triangleq \infty$. Formally,

$$B_{\Pi^*}^- \triangleq \inf\{B \ge 0 : \Pi^* \subseteq C_B\}$$

Next, we define notation and conditions used in the statement of the theorem. Given a PMV-instability setting, Π, B, and n, define

$$d_0 = \dim(C_0), d_\infty = \dim(C_\infty), \text{ and } d_\Delta = d_\infty - d_0,$$

where $\dim(C_0)$ is the *dimension* of C_0, which is the dimension of the minimal affine space that contains C_0. We also define the following five conditions:

κ_1	κ_2	κ_3
$\mathcal{U}_{n,B} = \emptyset$	$\mathrm{CH}(\Pi) \cap C_\infty = \emptyset$	$\mathrm{CH}(\Pi) \cap C_0 = \emptyset$
κ_4	κ_5	
$\mathrm{CH}(\Pi) \subseteq C_\infty$	$\mathrm{CH}(\Pi) \subseteq C_0$	

Recall that $\mathrm{CH}(\Pi)$ is the convex hull of Π. Because $C_0 \subseteq C_\infty$, $d_\Delta \ge 0$. Also notice that κ_2 implies κ_3, or equivalently, $\neg\kappa_3$ implies $\neg\kappa_2$. Similarly, κ_4 implies κ_5, or equivalently, $\neg\kappa_5$ implies $\neg\kappa_4$.

Theorem 2. (Max-Semi-Random PMV-instability, $B = O(n)$). *Given any $q \in \mathbb{N}$, any closed and strictly positive Π over $[q]$, and any PMV-instability setting $\mathcal{S} = \langle \mathcal{H}_S, \mathcal{H}_T, \mathbb{O}, \vec{c}\rangle$, any $C_2 > 0$ and $C_3 > 0$ with $C_2 < B_{CH(\Pi)} < C_3$, any $n \in \mathbb{N}$, and any $B \ge 0$,*

$\sup_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}\right) =$

Name	Likelihood		Condition
0 case	0		κ_1
exp case	$\exp(-\Theta(n))$		$\neg\kappa_1 \wedge \kappa_2$
PT-$\Theta(\sqrt{n})$	$\Theta\left(\frac{\min\{B+1,\sqrt{n}\}^{d_\Delta}}{(\sqrt{n})^{q-d_0}}\right)$		$\neg\kappa_1 \wedge \neg\kappa_3$
PT-$\Theta(n)$	$\exp(-\Theta(n))$	*if* $B \le C_2 n$	*otherwise, i.e.,* $\neg\kappa_1 \wedge \neg\kappa_2 \wedge \kappa_3$
	$\Theta\left((\frac{1}{\sqrt{n}})^{q-d_\infty}\right)$	*if* $B \ge C_3 n$	

The full version of the theorem in Appendix C.3 also characterizes the min part. The $B+1$ in both sup and inf are introduced to handle the $B < 1$ case. For every $B \ge 1$, we have $\Theta(B+1) = \Theta(B)$. The proof can be found in Appendix E.2.

4 Applications

In this section, we present three applications of Theorem 1 (or more precisely, the first and second steps of its proof) that lead to concise and informative characterizations.

The overall approach. In light of the three steps of the proof of Theorem 1, we propose the following two-step Procedure 1 for characterizing a specific semi-random X for a specific voting rule r, which correspond to Step 1 and 3 above, respectively.

Procedure 1: Characterizing semi-random coalitional influence X under voting rule r

Step (i): Model X under r as a PMV-multi-instability setting $\mathcal{M}$ as done in Step 1 above.
Step (ii): Characterize the conditions and degree of polynomial for $\mathcal{M}$ by applying Theorem 2 or 3 (in Appendix C.4).

Step (i) is often easy, and for Theorems 4 and 5 below, the modeling is the same as in the proof of Lemma 1. The hardness of Step 2 is highly problem-dependent, and we see two potential difficulties in Step (ii): first, it is sometimes not easy to verify for which n, B, and PMV-instability problem, the 0 case of Theorem 2 does not happen; second, sometimes d_0, and d_Δ can be hard to characterize.

The first application is a matching lower bound for many commonly studied GSRs, whose definitions can be found in Appendix B.1. Recall that π_{uni} is the uniform distribution over $\mathcal{L}(\mathcal{A})$.

Theorem 4. *Let r be an integer positional scoring rule, STV, ranked pairs, Schulze, maximin, or Copeland with lexicographic tie-breaking for any fixed $m \geq 3$. For any closed and strictly positive Π with $\pi_{uni} \in CH(\Pi)$, any $X \in \{\text{CM}, \text{MoV}\}$, there exists $N > 0$ such that for any $n > N$ and any $B \geq 1$,*

$$\widetilde{X}^{\max}_{\Pi}(r, n, B) = \Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}\right)$$

The full statement of Theorem 4 (including other coalitional influence problems such as the ones defined in Appendix B.2) and its full proof can be found in Appendix G.1. The next example shows an application of Theorem 4 to Borda.

Example 5. *In the setting of Example 2, notice that $\pi_{uni} = \frac{1}{2}(\pi^1 + \pi^2)$, which means that $\pi_{uni} \in CH(\Pi)$. It follows from Theorem 4 that for all sufficiently large n, $\widetilde{\text{CM}}^{\max}_{\Pi}(Borda, n, 1) = \Theta\left(\frac{1}{\sqrt{n}}\right)$.*

Theorem 4 leads to the following corollary on IC, where $\Pi = \{\pi_{\text{uni}}\}$.

Corollary 1. *Let r be an integer positional scoring rule, STV, ranked pairs, Schulze, maximin, or Copeland with lexicographic tie-breaking for fixed $m \geq 3$. For any $X \in \{\text{CM}, \text{MoV}\}$, there exists $N > 0$ such that for any $n > N$ and any $B \geq 1$, $\Pr_{P\sim(\pi_{uni})^n}(X(r, P, B)) = \Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}\right)$.*

The second application is an upper bound for all GSRs and all closed and strictly positive Π.

Theorem 5. *Let r denote any GSR with fixed $m \geq 3$. For any closed and strictly positive Π, any $X \in \{\text{CM}, \text{MoV}\}$, any n, and any $B \geq 1$,*

$$\widetilde{X}^{\max}_{\Pi}(r, n, B) = O\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}\right)$$

The full proof can be found in Appendix G.2. Theorem 5 immediately extends all previous $O\left(\frac{1}{\sqrt{n}}\right)$ upper bound on CM for a single manipulator ($B = 1$) discussed in Section 1.2 to any coalition size $B \geq 1$, because all rules studied in these works are GSRs.

The third application studies a new notion of coalitional manipulation that aims at making the loser win under integer positional scoring rules. For any positional scoring rule, the loser is the alternative with the minimum total score.

Definition 7. *Given any integer positional scoring rule $r_{\vec{s}}$ with lexicographic tie-breaking, for any profile P and any $B \geq 0$, we define $\text{CML}(r_{\vec{s}}, P, B) = 1$ if and only if a coalition of no more than B voters have incentive to misreport their preferences to make the loser under P win.*

Clearly, under veto, no coalition of voters have incentive to misreport their preferences to make the loser win, i.e., $\text{CML}(\text{veto}, P, B) = 0$ for all P and B.

Theorem 6. *Let $r_{\vec{s}}$ be an integer positional scoring rule with lexicographic tie-breaking for fixed $m \geq 3$ that is different from veto. For any closed and strictly positive Π with $\pi_{uni} \in CH(\Pi)$, there exists $N > 0$ and $B^* > 0$ such that for any $n > N$ and any $B \geq B^*$,*

$$\widetilde{\text{CML}}^{\max}_{\Pi}(r_{\vec{s}}, n, B) = \Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}^{m-1}\right)$$

The proof can be found in Appendix G.3.

5 Summary and Future Work

We extend previous studies on average-case likelihood of coalitional manipulation in elections in three aspects: (1) a more general and realistic semi-random model, (2) many other coalitional influence problems, and (3) arbitrary coalition size, by taking a polyhedral approach and developing and applying the PMV-(multi)-instability theorems (Theorem 2 and 3). While we do not think that results in this paper are the final answers to the question in Introduction, we do believe that they are a non-trivial step forward, because they address long-standing open questions and expand the scope of previous work to more general and realistic settings.

Moving forward, we see three natural directions for future work. **More realistic models:** as discussed in Section 2, the semi-random model in this paper assumes independent noises across agents and a strictly positive Π. A natural question is how to relax these constraints to build a more realistic, and still tractable, model. **Stronger theorems:** as discussed after Theorem 2, characterizing the constants in the bounds is a natural open question. Additionally, as discussed in the Introduction, Theorem 5 poses a challenge to designing natural rules with lower likelihood of coalitional influence, or improving the $\Omega(\frac{1}{n^{-67}})$ lower bound (and extending it to $B \geq 2$). **More applications:** how to develop informative characterizations in the spirit of Theorem 1, as done in Section 4, for other rules, other coalition influence problems, or other applications such as matching, resource allocation, fair division, judgement aggregation, are important and challenging tasks.

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Contents

A Related Work with More Details

Likelihood of coalitional manipulability: upper bounds under IC. Pattanaik [51] proposed to study the likelihood of strategic voting by single manipulator and conjectured that the likelihood is smaller in larger elections. Pazner and Wesley [53] proved that the likelihood of single-voter manipulability goes to 0 as $n \to \infty$ under plurality. They also noted that the results can be extended to any coalition of n^α manipulators, where $0 \leq \alpha < 1/2$. Peleg [54] proved that the likelihood of single-voter manipulability under any *representable voting system*, which includes positional scoring rules, goes to 0 under any positive i.i.d. distributions. Peleg also considered coalitional manipulation problem, by noticing in footnote 2 that the result still hods for any coalition of $o(\sqrt{n})$ voters. Nitzan [48] demonstrated that the likelihood decreases when n is large under Borda, plurality, and range voting. Fristrup and Keiding [26] proved an $O\left(\frac{1}{\sqrt{n}}\right)$ rate of convergence for single-voter manipulability under plurality. Kim and Roush [35] proved that maixmin (a.k.a., Simpson's method) can be manipulated by a coalition of unlimited number of voters almost surely as $n \to \infty$. Slinko [64, 65, 66] proved an $O\left(\frac{1}{\sqrt{n}}\right)$ upper bound for a single manipulator under plurality with runoff, representable voting system, all voting rules based on the unweighted majority graphs, and Bucklin. Baharad and Neeman [4] proved an $O\left(\frac{1}{\sqrt{n}}\right)$ upper bound for a single manipulator under positional scoring rule, top cycle, and Copeland, when voters' preferences has small local correlations, which is more general than IC. Slinko [67] investigated the likelihood of coalitional manipulation by up to Cn^α manipulator for any fixed $0 \leq a < 1/2$, and proved an $O((\frac{1}{n})^{0.5-\alpha})$ upper bound for any positional scoring rule with strictly decreasing scores. Maus et al. [43] characterized the least manipulable rule by a single manipulator among tops-only, anonymous, and surjective choice rules, to be the unanimity rules with status quo. Pritchard and Wilson [57] showed that, the likelihood for a coalition of $v\sqrt{n}$ manipulators to succeed under a positional scoring rule is a function of v, and provided an algorithm based on integer linear program to compute the minimum coalition size.

Likelihood of coalitional manipulability: lower bounds under IC. Slinko [64] proved an $\Omega(\frac{1}{\sqrt{n}})$ lower bound under plurality for a single manipulator. A quantitative Gibbard-Satterthwaite theorem was proved for $m = 3$ by Friedgut et al. [25], and was subsequently developed in [18, 33, 80], and the general case was proved by Mossel and Racz [47], which implies that any voting rule that is constantly away from any dictatorships, the likelihood of single-voter manipulability under IC is $\Omega\left(\frac{1}{n^{67}m^{166}}\right)$, i.e., $\Omega(\frac{1}{n^{67}})$ for any fixed m.

Likelihood of coalitional manipulability: simulations. Beyond theoretical work, there is also a large literature on comparing the empirical coalitional manipulability of commonly-studied voting rules, mostly by computer simulations [1–3, 11, 31, 34, 56, 69]. These works confirm that the likelihood for a large election to be manipulable by a single manipulator is low.

Likelihood of coalitional manipulability under other distributions. As discussed above, the convergence-to-0 result for representable voting system in [54] works for any i.i.d. distributions, and the $O(1/\sqrt{n})$ upper bound for positional scoring rule, top cycle, and Copeland by Baharad and Neeman [4] works for distributions with small local correlations. Procaccia and Rosenschein [58] proved that for weighted voters whose preferences are generated independently, positional scoring rules cannot be manipulated by $o(\sqrt{n})$ manipulators almost surely as $n \to \infty$. Xia and Conitzer [79] proved that for a large of class of voting rules and i.i.d. distributions that satisfy certain conditions, a coalition of $O(n^\alpha)$ manipulators is powerless when $\alpha < \frac{1}{2}$, as the likelihood for them to succeed is $O(n^{\alpha-0.5})$; and they are powerful when $\alpha > \frac{1}{2}$, as the likelihood is lower bounded by $1 - \exp(-\Omega(n))$ in such case. Mossel et al. [46] illustrates a smooth transition from powerlessness to powerfulness for a coalition of $c\sqrt{n}$ manipulators with variable c. Xia [75] proved that for a large class of influence problems including coalitional manipulation, under i.i.d. distributions, with probability that goes to 1 the number of voters needed is 0, $\Theta(\sqrt{n})$, $\Theta(n)$, or impossible. The paper does not characterize the likelihood for each case. Durand et al. [20] used Condorcification to decrease coalitional manipulable profiles, reduces coalitional manipulability under every probability distribution.

There is a large literature on the likelihood of coalitional manipulability under the *Impartial Anonymous Culture (IAC)*, which assumes that each histogram happens equally likely and resembles the flat Dirichlet distribution, based on theoretical analysis [22, 23, 32, 37–39, 61, 68, 74] and computer simulations [31, 40, 56]. Both IAC and IC are mainly of theoretical interest and *"are poor proxies of political electorates"* [50].

Other coalitional influence problems. The *margin of victory (MoV)* of a profile is the smallest coalition of voters who can change the winner by casting different votes (regardless of their preferences). MoV measures the stability of elections and provides an upper bound on CM and some previous proofs of upper bounds on CM are done for MoV, such as [64, 79]. Pritchard and Slinko [55] proved that for any positional scoring rule, the expected margin of victory under IC is $\Theta(\sqrt{n})$ and characterized the voting rules with maximum expected MoV under IC. Results in [46, 75] discussed above also apply to MoV. Brill et al. [9] studies the distribution of MoV for some tournament rules, under a probability distribution on tournament graphs, where the direction of each edge is drawn independently and uniformly.

The likelihood of MoV = 1 has been used to measure the *decisiveness* of voting, sometimes called *voting power*, which plays an important role in the paradox of voting [19] and in definitions of power indices in cooperative game theory. For two alternatives under the plurality rule, the voting power is equivalent to the likelihood of ties [30] (see, e.g., [77] for a semi-random analysis on the likelihood of ties and references therein). In general, the two problems are different, for example as shown in Example 6 and 7.

Beyond CM and MoV, there is a large body of work on the computational complexity of other types of coalition influence problems that based on vote operations, such as constructive/destructive control (making a designated alternative win/lose) by adding/deleting votes, and bribery (different changes in votes have different costs and a total budget is given). See [21] for a recent survey. Little work has been done to analyze their likelihood of success, except [75], which as discussed above, does not provide an accurate characterization.

B Extra Preliminaries

B.1 Other commonly studied voting rules

Weighted Majority Graphs. For any (fractional) profile P and any pair of alternatives a, b, let $P[a \succ b]$ denote the total weight of votes in P where a is preferred to b. Let $\text{WMG}(P)$ denote the *weighted majority graph* of P, whose vertices are $\mathcal{A}$ and whose weight on edge $a \to b$ is $w_P(a, b) = P[a \succ b] - P[b \succ a]$. Sometimes a distribution π over $\mathcal{L}(\mathcal{A})$ is viewed as a fractional profile, where for each $R \in \mathcal{L}(\mathcal{A})$ the weight on R is $\pi(R)$. In this case we let $\text{WMG}(\pi)$ denote the weighted majority graph of the fractional profile represented by π.

A voting rule is said to be *weighted-majority-graph-based (WMG-based)* if its winners only depend on the WMG of the input profile. In this paper we consider the following commonly studied WMG-based rules.

- **Copeland.** The Copeland rule is parameterized by a number $0 \le \alpha \le 1$, and is therefore denoted by Copeland_α, or Cd_α for short. For any fractional profile P, an alternative a gets 1 point for each other alternative it beats in their head-to-head competition, and gets α points for each tie. Copeland_α chooses all alternatives with the highest total score as the winners.
- **Maximin.** For each alternative a, its min-score is defined to be $\min_{b \in \mathcal{A}} w_P(a, b)$. Maximin, denoted by MM, chooses all alternatives with the max min-score as the winners.
- **Ranked pairs.** Given a profile P, an alternative a is a winner under ranked pairs (denoted by RP) if there exists a way to fix edges in $\text{WMG}(P)$ one by one in a non-increasing order w.r.t. their weights (and sometimes break ties), unless it creates a cycle with previously fixed edges, so that after all edges are considered, a has no incoming edge. Ties between edges are broken lexicographically. For example, if $1 \to 2$ and $2 \to 3$ have the same weight, then $1 \to 2$ is chosen first. If $1 \to 2$ and $1 \to 3$ have the same weight, then $1 \to 2$ is chosen first.
- **Schulze.** For any directed path in the WMG, its strength is defined to be the minimum weight on any single edge along the path. For any pair of alternatives a, b, let $s[a, b]$ be the highest weight among all paths from a to b. Then, we write $a \succeq b$ if and only if $s[a, b] \ge s[b, a]$, and [63] proved that the strict version of this binary relation, denoted by $\succ$, is transitive. The Schulze rule, denoted by Sch, chooses all alternatives a such that for all other alternatives b, we have $a \succeq b$.

STV with lexicographic tie-breaking mechanism. The (single-winner) STV with lexicographic tie-breaking chooses winners in $m - 1$ rounds. In each round, the *loser* of plurality under lexicographic tie-breaking is removed from the election. We note that this rule is different from first computing STV winners under *parallel universe tie-breaking* [14] and then breaking ties among the co-winners.

Plurality with runoff. The plurality with runoff rule with lexicographic tie-breaking, denoted by Pro, chooses the winner in two rounds. In the first round, the two alternatives with highest plurality scores are chosen (ties are broken lexicographically), and all other alternatives are removed. In the second round, the majority rule with lexicographic tie-breaking is applied to choose the winner.

Proposition 1. *Any representable voting system with lexicographic tie-breaking is a GSR.*

Proof. The proof is similar to the proof that shows positional scoring rules with lexicographic tie-breaking are GSRs illustrated in Example 1. Formally, we define the following score difference vector that is similar to the score difference vector defined for positional scoring rules [76].

Definition 8 (**Score difference vector for representable voting system**). *For any scoring function $s : \mathcal{L}(\mathcal{A}) \times \mathcal{A} \to \mathbb{R}_{\ge 0}$ and any pair of different alternatives a, b, let $\text{Score}^s_{a,b}$ denote the $m!$-dimensional vector indexed by rankings in $\mathcal{L}(\mathcal{A})$: for any $R \in \mathcal{L}(\mathcal{A})$, the R-element of $\text{Score}^s_{a,b}$ is $s(R, a) - s(R, b)$.*

Let $K = \binom{m}{2}$ and the hyperplanes are score difference vectors $\{\text{Score}^s_{a,b} : a \in \mathcal{A}, b \in \mathcal{A}, a \ne b\}$. For any profile P, $\text{Sign}_{\vec{H}}(\text{Hist}(P))$ contains information about the comparisons of total scores of all pairs of alternatives, from which g chooses a winner and applies the tie-breaking mechanism when needed. □

B.2 Other commonly studied coalitional influence problems

In the *constructive control by adding votes (CCAV)* (respectively, *destructive control by adding votes (DCAV)*) problem, we are given a distinguished alternative d, and we let $\text{CCAV}_d(r, P, B) = 1$ (respectively, $\text{DCAV}_d(r, P, B) = 1$), if there exists a preference profile P^* with $|P^*| \le B$ such that $r(P + P^*) = \{d\}$ (respectively, $r(P + P^*) \ne \{d\}$).

In the *constructive control by deleting votes (CCDV)* (respectively, *destructive control by deleting votes (DCDV)*) problem, we are given a distinguished alternative d, and we let $\text{CCDV}_d(r, P, B) = 1$ (respectively, $\text{DCDV}_d(r, P, B) = 1$), if there exists $P' \subseteq P$ with $|P'| \le B$ such that $r(P - P^*) = \{d\}$ (respectively, $r(P - P^*) \ne \{d\}$).

For convenience, we let Control denote the control problems introduced above, and let e-Control denote their effective variants, formally defined as follows.

Definition 9 (**Effective control problems**). *Define*

$$\text{E-CONTROL} = \{\text{E-CCAV}, \text{E-CCDV}, \text{E-DCAV}, \text{E-DCDV}\} \textit{ and}$$

$$\text{CONTROL} = \{\text{CCAV}, \text{CCDV}, \text{DCAV}, \text{DCDV}\}$$

C Materials for Section 3

C.1 Constructive/Destructive Generalized Bribery with Anonymous Prices

In this section, we define two large classes of bribery problems that include some commonly studied control problems as special cases.

Definition 10. *A* constructive generalized bribery with anonymous prices *problem is denoted by* $\text{CB}_{d,\vec{c}}(r, P, B)$*, where* r *is a voting rule,* P *is a profile,* a *is a distinguished alternative,* $\vec{c} > \vec{0}$ *is a strictly positive cost vector, where each component is indexed by a pair* $(R, R') \in (\mathcal{L}(\mathcal{A}) \cup \{\emptyset\}) \times (\mathcal{L}(\mathcal{A}) \cup \{\emptyset\})$ *that represents the price for the briber to convert an* R *vote to an* R' *vote, and* $B \geq 0$ *is the total budget. We are asked whether the briber can make a win by changing the votes in the profile under the budget constraint* B*—if so then we let* $\text{CB}_{d,\vec{c}}(r, P, B) = 1$*, otherwise we let* $\text{CB}_{d,\vec{c}}(r, P, B) = 0$.

Destructive bribery with anonymous price *problem, denoted by* $\text{DB}_{d,\vec{c}}(r, P, B)$*, is defined similarly, and the only difference is that the goal of the briber is to make* a not *the winner.*

Specifically, when $R = \emptyset$, performing an (R, R') bribery is effectively the same as adding an R' vote to P; and if $R' = \emptyset$, performing an (R, R') bribery is effectively the same as removing an R vote from P. Moreover, we allow the price of an (R, R') operation to be ∞, which means that this operation is not allowed in the problem.

For each constructive/destructive control/bribery problem, we also study its "effective" variant, which requires that the influencers' goal is not achieved in the original profile. We will add "E-" to the name to denote this variant. For example, $\text{E-CB}_{d,\vec{c}}(r, P, B) = 1$ if $r(P) \neq \{a\}$ and a can be made the winner under budget B.

Proposition 2. CCAV_d *and* CCDV_d *are special cases of* $\text{CB}_{d,\vec{c}}$. E-CCAV_d *and* E-CCDV_d *are special cases of* $\text{E-CB}_{d,\vec{c}}$. DCAV_d *and* DCDV_d *are special cases of* $\text{DB}_{d,\vec{c}}$. E-DCAV_d *and* E-DCDV_d *are special cases of* $\text{E-DB}_{d,\vec{c}}$.

Proof. It is not hard to verify that CCAV_d (respectively, DCAV_d) is equivalent to $\text{CB}_{d,\vec{c}}$ (respectively, $\text{DB}_{d,\vec{c}}$), where for any $(R, R') \in (\mathcal{L}(\mathcal{A}) \cup \{\emptyset\}) \times (\mathcal{L}(\mathcal{A}) \cup \{\emptyset\})$, the (R, R') component of $\vec{c}$, denoted by $[\vec{c}\,]_{(R,R')}$, is $\begin{cases} 1 & \text{if } R = \emptyset \\ 0 & \text{otherwise} \end{cases}$.

CCDV_d (respectively, DCDV_d) is equivalent to $\text{CB}_{d,\vec{c}}$ (respectively, $\text{DB}_{d,\vec{c}}$), where for any $(R, R') \in (\mathcal{L}(\mathcal{A}) \cup \{\emptyset\}) \times (\mathcal{L}(\mathcal{A}) \cup \{\emptyset\})$,

$$[\vec{c}\,]_{(R,R')} = \begin{cases} 1 & \text{if } R' = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

The proofs for E-variants are similar. □

C.2 Full Version of Theorem 1 and Its Proof

Theorem 1. (Semi-Random Coalitional Influence, Full Version). *Let* r *denote any GSR with fixed* $m \geq 3$*. For any closed and strictly positive* Π*, any coalitional influence problem* $X \in \{\text{CM}, \text{MoV}, \text{CB}_{d,\vec{c}}(r, P, B), \text{DB}_{d,\vec{c}}(r, P, B)\}$*, there exists a constant* $C_1 > 0$ *so that for any* $n \in \mathbb{N}$ *and any* $B \geq 0$ *with* $B \leq C_1 n$*, there exists* $\{d_0^{\max}, d_0^{\min}, d_\Delta^{\max}, d_\Delta^{\min}\} \subseteq [m!]$ *such that*

$$\widetilde{X}_\Pi^{\max}(r, n, B) \textit{ is } 0, \exp(-\Theta(n)), \textit{ or } \Theta\left(\frac{\min\{B+1, \sqrt{n}\}^{d_\Delta^{\max}}}{(\sqrt{n})^{m! - d_0^{\max}}}\right), \textit{ and}$$

$$\widetilde{X}_\Pi^{\min}(r, n, B) \textit{ is } 0, \exp(-\Theta(n)), \textit{ or } \Theta\left(\frac{\min\{B+1, \sqrt{n}\}^{d_\Delta^{\min}}}{(\sqrt{n})^{m! - d_0^{\min}}}\right)$$

Proof. We continue with Step 2 and 3.

C.3 Step 2 of The Proof: The PMV-Instability Theorem

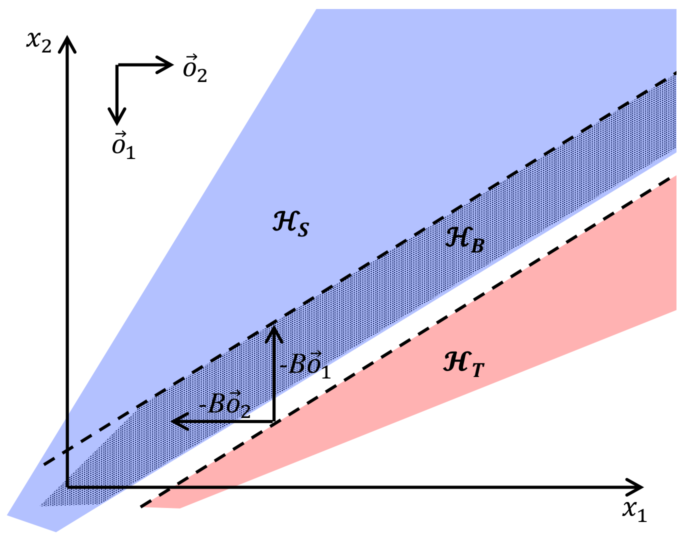
Let us start with some high-level intuitions for solving the PMV-instability problem (Definition 4) to motivate the statement and proof of Theorem 2.

Intuition. If $\mathcal{U}_{n,B} = \emptyset$, then the (max- or min-) semi-random instability is 0, and we will refer to this as the **0 case**. Suppose the 0 case does not hold, i.e., $\mathcal{U}_{n,B} \neq \emptyset$, then we will adopt an approximation of $\Pr(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B})$ based on the following two approximations.

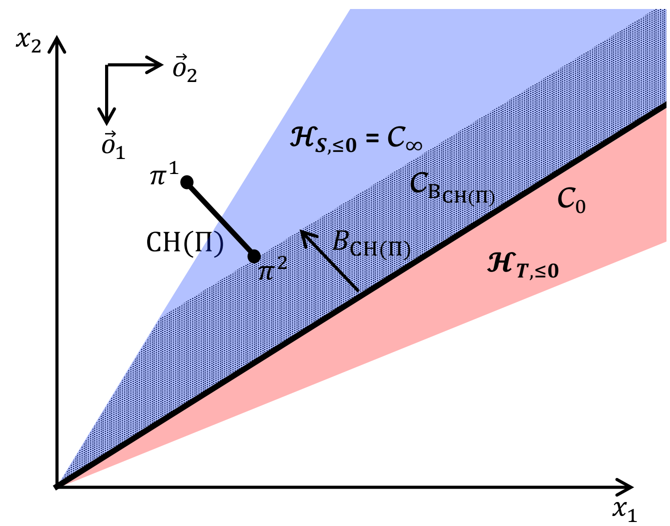
- First, let us *pretend* that all integrality constraints in the linear system (like the one in Example 3) are relaxed. That is, let us pretend that $\vec{X}_{\vec{\pi}}$ can take non-integer values, and the vote operation vector $\vec{o}$ can be fractional (but still need to be non-negative). For simplicity, assume that $\vec{c} = \vec{1}$. Then, the possible changes to the histogram as the result of a unit of budget is characterized by $\text{CH}(\mathbb{O})$, where $\text{CH}(\cdot)$ represents the convex hull. It follows that $\mathcal{U}_{n,B}$ can be approximated by a polyhedron $\mathcal{H}_B$, which is the intersection of $\mathcal{H}_\text{S}$ and the Minkowski addition of $\mathcal{H}_\text{T}$ and $\bigcup_{0\le b\le B} -b\cdot \text{CH}(\mathbb{O})$. See Figure 2 (a) for an illustration of $\mathcal{H}_B$ in the shaded area, where $\mathbb{O} = \{\vec{o}_1, \vec{o}_2\}$. Notice that $\mathcal{H}_B$ is in the $-\text{CH}(\mathbb{O})$ direction of $\mathcal{H}_\text{T}$.
- Second, let us *pretend* that $\vec{X}_{\vec{\pi}}$ is distributed as a $(q-1)$-dimensional Gaussian distribution $\mathcal{N}_{\vec{\pi}}$ (whose mean is $\sum_{j=1}^n \pi_j \in n\cdot\text{CH}(\Pi)$) in the hyperplane $\{\vec{x} : \vec{x}\cdot\vec{1} = n\}$. This approximation is justified by various multi-variable central limit theorems, e.g., [6, 16, 17, 59, 73].

With these approximations, we adopt the following approximation of $\Pr(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B})$:

$$\Pr(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}) \approx \Pr(\mathcal{N}_{\vec{\pi}} \in \mathcal{H}_B) \tag{3}$$



(a) $\mathcal{H}_B$.



(b) $B_{\text{CH}(\Pi)}$ and $C_{B_{\text{CH}(\Pi)}}$.

Figure 2: Illustrations of some notation.

Now, let us take a look at the max-semi-random PMV instability in light of the approximation in (3). Because the probability mass of $\mathcal{N}_{\vec{\pi}}$ is mostly centered around an $O(\sqrt{n})$ neighborhood of its mean, to maximize $\Pr(\mathcal{N}_{\vec{\pi}} \in \mathcal{H}_B)$, the adversary aims at choosing $\vec{\pi}$ so that the "volume" of the intersection of an $O(\sqrt{n})$ neighborhood of $\sum_{j=1}^n \pi_j$ and $\mathcal{H}_B$ is as large as possible.

When $B = O(\sqrt{n})$, it turns out that $\mathcal{H}_B$ is close to $\mathcal{H}_{\text{S},\le 0}\cap\mathcal{H}_{\text{T},\le 0}$, where for any polyhedron $\mathcal{H} \triangleq \{\vec{x} : \mathbf{A}\times(\vec{x})^\top \le (\mathbf{b})^\top\}$, $\mathcal{H}_{\le 0} \triangleq \{\vec{x} : \mathbf{A}\times(\vec{x})^\top \le \left(\vec{0}\right)^\top\}$ denotes its *characteristic cone*, also known as the *recess cone*. Therefore, if $\text{CH}(\Pi)\cap\mathcal{H}_{\text{S},\le 0}\cap\mathcal{H}_{\text{T},\le 0} = \emptyset$, then the mean of $\vec{X}_{\vec{\pi}}$ is $\Theta(n)$ away from $\mathcal{H}_{\text{S},\le 0}\cap\mathcal{H}_{\text{T},\le 0}$, which implies that the likelihood is (exponentially) small due to straightforward applications of Hoeffding's inequality. We call this case the **exponential case**.

When $B = O(\sqrt{n})$ and $\text{CH}(\Pi)\cap\mathcal{H}_{\text{S},\le 0}\cap\mathcal{H}_{\text{T},\le 0} \ne \emptyset$, the adversary can choose $\vec{\pi}\in\Pi^n$ so that the mean of $\vec{X}_{\vec{\pi}}$ is either in $\mathcal{H}_{\text{S},\le 0}\cap\mathcal{H}_{\text{T},\le 0}$ or is $O(1)$ away, which means that the likelihood is large. In this case, there is a **phase transition at $B = \Theta(\sqrt{n})$**, as it will be shown that the likelihood reaches its (asymptotic) max at $B = \Theta(\sqrt{n})$.

When $B = \Theta(n)$, again, the adversary aims at choosing $\vec{\pi}\in\Pi^n$ so that the mean of $\vec{X}_{\vec{\pi}}$ is close to $\mathcal{H}_B$. Let $B_{\text{CH}(\Pi)}$ denote the smallest budget so that a "pseudo-conic" approximation to $\mathcal{H}_{B_{\text{CH}(\Pi)}}$, denoted by C_B and is formally defined below in (4), touches $\text{CH}(\Pi)$, which is the convex hull of Π. See Figure 2 (b) for an illustration of $B_{\text{CH}(\Pi)}$ and $C_{B_{\text{CH}(\Pi)}}$ (the shaded area). It is then expected that when $B < B_{\text{CH}(\Pi)}\cdot n$, the max-semin-random instability whould be small, and when $B > B_{\text{CH}(\Pi)}\cdot n$, the max-semin-random instability whould be large. In other words, the semi-random stability has a **phase transition at $B = \Theta(n)$**.

Nevertheless, characterizing the conditions and likelihood for each case is still challenging, as the approximation above is only meant to provide a qualitative intuition. Existing multi-variate central limit theorems are often too coarse due to an $\Omega(\frac{1}{\sqrt{n}})$ error, as discussed in [77].

Notation. Let us define some notation to formalize the intuitions discussed above. For any budget $B \geq 0$, we let $\mathcal{H}_B$ denote the relaxation of $\mathcal{U}_{n,B}$ by removing the size constraint and the integrality constraints on $\vec{x}$ and on $\vec{o}$. Recall that $\vec{o}$ still needs to be non-negative. That is,

$$\mathcal{H}_B \triangleq \left\{\vec{x} \in \mathcal{H}_{\mathrm{S}} : \exists \vec{o} \in \mathbb{R}_{\geq 0}^{|\mathbb{O}|} \text{ s.t. } \vec{c} \cdot \vec{o} \leq B \text{ and } \vec{x} + \vec{o} \times \mathbf{O} \in \mathcal{H}_{\mathrm{T}}\right\}$$

For example, Figure 2 (a) illustrates $\mathcal{H}_B$ in the shaded area. $\mathcal{H}_B$ is a polyhedron because it is the intersection of $\mathcal{H}_{\mathrm{S}}$ and the Minkowski addition of $\mathcal{H}_{\mathrm{T}}$ and the following polyhedron Q_B:

$$Q_B \triangleq \left\{-\vec{o} \times \mathbf{O} : \vec{o} \in \mathbb{R}_{\geq 0}^{|\mathbb{O}|} \text{ and } \vec{c} \cdot \vec{o} \leq B\right\}$$

Specifically, with infinite budget ($B = \infty$), we have

$$\mathcal{H}_\infty = \left\{\vec{x} \in \mathcal{H}_{\mathrm{S}} : \exists \vec{o} \in \mathbb{R}_{\geq 0}^{|\mathbb{O}|} \text{ s.t. } \vec{x} + \vec{o} \times \mathbf{O} \in \mathcal{H}_{\mathrm{T}}\right\} = \mathcal{H}_{\mathrm{S}} \cap (\mathcal{H}_{\mathrm{T}} + Q_\infty)$$

For every $B \geq 0$, we define C_B to be the polyhedron that consists of all (possibly non-integer) vectors in $\mathcal{H}_{\mathrm{S},\leq 0}$ that can be manipulated to be in $\mathcal{H}_{\mathrm{T},\leq 0}$ by using (possibly non-integer) operations $\vec{o}$ under budget constraint B. That is,

$$C_B \triangleq \left\{\vec{x} \in \mathcal{H}_{\mathrm{S},\leq 0} : \exists \vec{o} \in \mathbb{R}_{\geq 0}^{|\mathbb{O}|} \text{ s.t. } \vec{c} \cdot \vec{o} \leq B \text{ and } \vec{x} + \vec{o} \times \mathbf{O} \in \mathcal{H}_{\mathrm{T},\leq 0}\right\} \tag{4}$$

It is not hard to verify that $C_B = \mathcal{H}_{\mathrm{S},\leq 0} \cap (\mathcal{H}_{\mathrm{T},\leq 0} + Q_B)$ and C_B can be viewed as a "pseudo-conic" approximation to $\mathcal{H}_B$, as C_B is defined based on the characteristic cones of $\mathcal{H}_{\mathrm{S}}$ and $\mathcal{H}_{\mathrm{T}}$, though C_B itself may not be a cone. Specifically, C_0 and C_∞ will play a central role in Theorems 2 and 3. It is not hard to verify that

$$C_0 \triangleq \mathcal{H}_{\mathrm{S},\leq 0} \cap \mathcal{H}_{\mathrm{T},\leq 0}, \text{ and}$$
$$C_\infty \triangleq \left\{\vec{x} \in \mathcal{H}_{\mathrm{S},\leq 0} : \exists \vec{o} \in \mathbb{R}_{\geq 0}^{|\mathbb{O}|} \text{ s.t. } \vec{x} + \vec{o} \times \mathbf{O} \in \mathcal{H}_{\mathrm{T},\leq 0}\right\} = \mathcal{H}_{\mathrm{S},\leq 0} \cap \left(\mathcal{H}_{\mathrm{T},\leq 0} + Q_\infty\right) \tag{5}$$

Figure 2 (b) illustrates C_0 (which is a line) and C_∞ (which is the same as $\mathcal{H}_{\mathrm{S},\leq 0}$). Both C_0 and C_∞ are polyhedral cones, because the intersection Minkowski addition of two polyhedral cones is a polyhedral cone. Notice that when $B \notin \{0, \infty\}$, C_B may not be a cone.

For any set $\Pi^* \subseteq \mathbb{R}^q$, let $B_{\Pi^*} \in \mathbb{R}$ to be the minimum budget B such that the intersection of Π^* and C_B is non-empty. If no such B exists (i.e., $\Pi^* \cap C_\infty = \emptyset$), then we let $B_{\Pi^*} \triangleq \infty$. Formally,

$$B_{\Pi^*} \triangleq \inf\{B^* \geq 0 : \Pi^* \cap C_B \neq \emptyset\} \tag{6}$$

Figure 2 (b) illustrates $B_{\mathrm{CH}(\Pi)}$ and $C_{B_{\mathrm{CH}(\Pi)}}$ in the shaded area, where $\Pi = \{\pi^1, \pi^2\}$ and $\mathrm{CH}(\Pi)$ is the *convex hull* of Π, which is the line segment between π^1 and π^2 in this case.

For any set $\Pi^* \subseteq \mathbb{R}^q$, we define $B^-_{\Pi^*} \in \mathbb{R}$ to be the minimum budget B such that Π^* is completely contained in C_B. If no such B exists, then we let $B^-_{\Pi^*} \triangleq \infty$. Formally,

$$B^-_{\Pi^*} \triangleq \inf\{B \geq 0 : \Pi^* \subseteq C_B\} \tag{7}$$

For example, $B^-_{\mathrm{CH}(\Pi)} = \infty$ in Figure 2 (b), because no matter how large B is, $C_B \subseteq \mathcal{H}_{\mathrm{S},\leq 0}$, and $\mathcal{H}_{\mathrm{S},\leq 0}$ does not contain all vectors in $\mathrm{CH}(\Pi)$.

Next, we define notation and conditions used in the statement of the theorem.

Definition 11. *Given a PMV-instability setting, Π, B, and n, define*

$$d_0 = \dim(C_0), d_\infty = \dim(C_\infty), \text{ and } d_\Delta = d_\infty - d_0,$$

where $\dim(C_0)$ *is the* dimension *of* C_0*, which is the dimension of the minimal affine space that contains* C_0*. We also define the following five conditions:*

κ_1	κ_2	κ_3	κ_4	κ_5
$\mathcal{U}_{n,B} = \emptyset$	$CH(\Pi) \cap C_\infty = \emptyset$	$CH(\Pi) \cap C_0 = \emptyset$	$CH(\Pi) \subseteq C_\infty$	$CH(\Pi) \subseteq C_0$

Recall that $\mathrm{CH}(\Pi)$ is the convex hull of Π. Because $C_0 \subseteq C_\infty$, $d_\Delta \geq 0$. Also notice that κ_2 implies κ_3, or equivalently, $\neg\kappa_3$ implies $\neg\kappa_2$. Similarly, κ_4 implies κ_5, or equivalently, $\neg\kappa_5$ implies $\neg\kappa_4$.

Theorem 2 (**Semi-Random PMV-Instability**). *Given any $q \in \mathbb{N}$, any closed and strictly positive Π over $[q]$, and any PMV-instability setting $\mathcal{S} = \langle \mathcal{H}_S, \mathcal{H}_T, \mathbb{O}, \vec{c}\rangle$, any $C_2 > 0$ and $C_3 > 0$ with $C_2 < B_{CH(\Pi)} < C_3$, any $n \in \mathbb{N}$, and any $B \geq 0$,*

$\sup_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}\right) =$

Name	**Likelihood**		**Condition**
0 case	0		κ_1
exp case	$\exp(-\Theta(n))$		$\neg\kappa_1 \wedge \kappa_2$
PT-$\Theta(\sqrt{n})$	$\Theta\left(\dfrac{\min\{B+1, \sqrt{n}\}^{d_\Delta}}{(\sqrt{n})^{q-d_0}}\right)$		$\neg\kappa_1 \wedge \neg\kappa_3$
PT-$\Theta(n)$	$\exp(-\Theta(n))$	*if $B \leq C_2 n$*	*otherwise, i.e.,* $\neg\kappa_1 \wedge \neg\kappa_2 \wedge \kappa_3$
	$\Theta\left((\frac{1}{\sqrt{n}})^{q-d_\infty}\right)$	*if $B \geq C_3 n$*	

For any $C_2^- < B^-_{CH(\Pi)} < C_3^-$*, any* $n \in \mathbb{N}$*, and any* $B \geq 0$,

$\inf_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}\right) =$

Name	**Likelihood**		**Condition**
0 case	0		κ_1
exp case	$\exp(-\Theta(n))$		$\neg\kappa_1 \wedge \kappa_4$
PT-$\Theta(\sqrt{n})$	$\Theta\left(\frac{\min\{B+1, \sqrt{n}\}^{d_\Delta}}{(\sqrt{n})^{q-d_0}}\right)$		$\neg\kappa_1 \wedge \neg\kappa_5$
PT-$\Theta(n)$	$\exp(-\Theta(n))$	*if* $B \leq C_2^- n$	*otherwise, i.e.,* $\neg\kappa_1 \wedge \neg\kappa_4 \wedge \kappa_5$
	$\Theta\left((\frac{1}{\sqrt{n}})^{q-d_\infty}\right)$	*if* $B \geq C_3^- n$	

The $B + 1$ in both sup and inf are introduced to handle the $B < 1$ case. For every $B \geq 1$, we have $\Theta(B + 1) = \Theta(B)$.

The four cases. Following the intuition presented at the beginning of this subsection, we call the first case of sup and inf in Theorem 2 the 0 *case*, the second case the *exponential case*, the third case the *phase transition at* $\Theta(\sqrt{n})$ *case* (PT-$\Theta(\sqrt{n})$ for short), and the last case, which contains two subcases, the *phase transition at* $\Theta(n)$ *case* (PT-$\Theta(n)$ for short). Notice that in each of the inf part, the likelihood is the same as its counterpart for the sup case, but the conditions and the threshold for the PT-$\Theta(n)$ case are different.
Figure 3 illustrates the (max- and min-) semi-random instability as a function of B for exp case, PT-$\Theta(\sqrt{n})$ case, and PT-$\Theta(n)$ case (for both sup and inf). Figure 4 (a) illustrates condition $\kappa_2 = [\text{CH}(\Pi) \cap C_\infty = \emptyset]$ for the exp case of sup. Figure 4 (b) illustrates condition $\neg\kappa_3 = [\text{CH}(\Pi) \cap C_0 \neq \emptyset]$ for the PT-$\Theta(\sqrt{n})$ case of sup. Figure 2 (b) illustrates condition $\neg\kappa_2 \wedge \kappa_3 = [\text{CH}(\Pi) \cap C_\infty \neq \emptyset] \wedge [\text{CH}(\Pi) \cap C_0 = \emptyset]$ for the PT-$\Theta(n)$ case of sup.

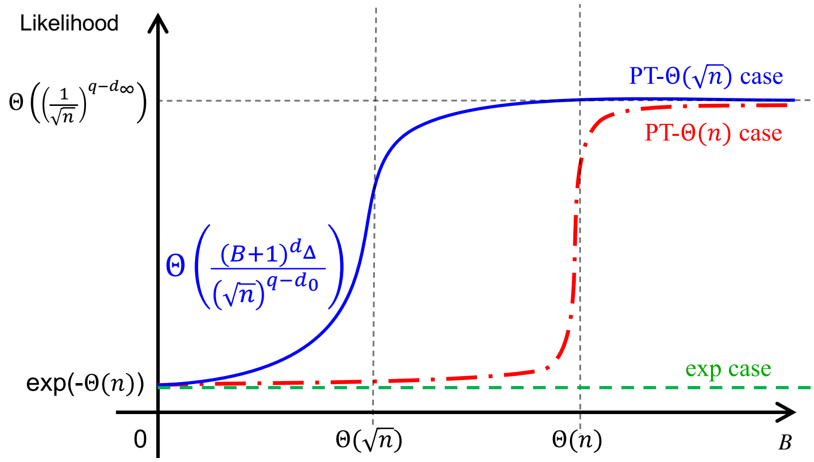
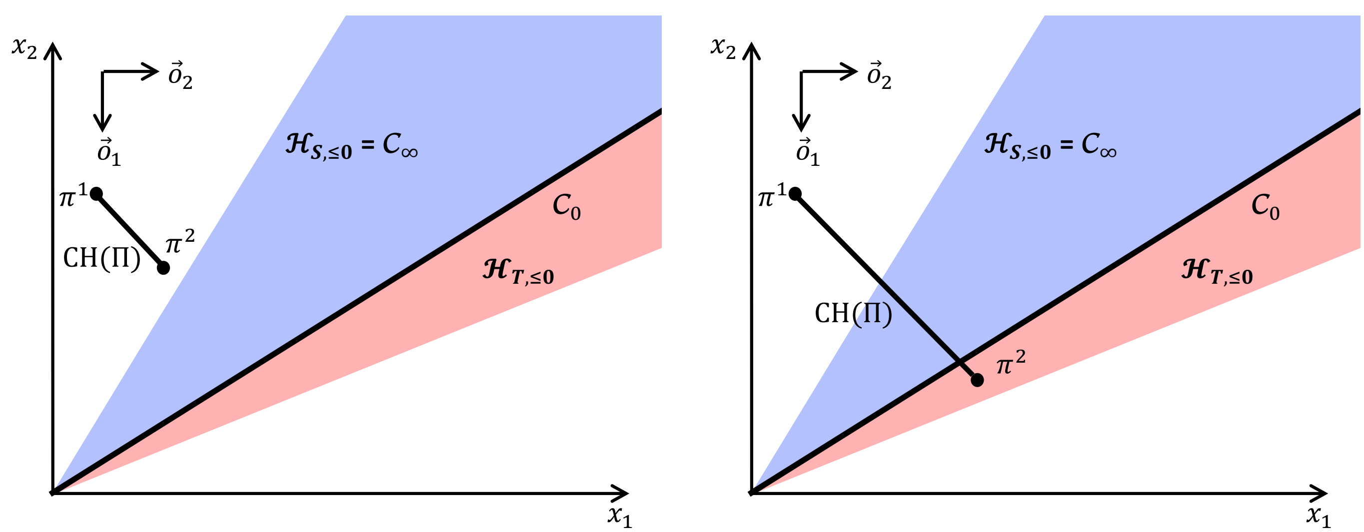


Figure 3: Illustration of Theorem 2. The x-axis is in log scale.



(a) The exp case of sup.

(b) The PT-$\Theta(\sqrt{n})$ case of sup.

Figure 4: Illustration of the exp case and phase-transition-at-$\Theta(\sqrt{n})$ case of sup.

Limitations and usefulness of Theorem 2. Theorem 2 has two major limitations. First, we were not able to characterize max-semi-random likelihood around $B_{\text{CH}(\Pi)} \cdot n$ (respectively, min-semi-random instability around $B^-_{\text{CH}(\Pi)} \cdot n$). Second, the constants in asymptotic bounds may be exponentially large in m.

Despite these limitations, we believe that Theorem 2 provides a general and useful tool for studying PMV-instability problems, because it converts the complicated PMV-instability problems, which involve reasoning about the likelihood of discrete events (about the PMV) that cannot be easily bounded by standard techniques, to deterministic geometric problems about $\text{CH}(\Pi)$, $\mathcal{C}_0$, $\mathcal{C}_\infty$, d_0, and d_∞. It provides an almost complete characterization of the PMV-instability problem, which can be easily applied to resolve long-standing open questions, e.g., in Corollary 1. Practically, $\dim_0$ and $\dim_\Delta$ can still be hard to characterize, but at least Theorem 2 provides a useful guideline about what to look for. See Section 4 for some examples.

Proof sketch of Theorem 2. At a high level, the proof follows after the intuitions presented in the beginning of this subsection. The hardest part is the proof of the (asymptotically tight) polynomial bounds in the PT-$\Theta(\sqrt{n})$ case. Take sup and $B = O(\sqrt{n})$ for example. To prove the polynomial upper bound, our proof can be viewed as upper-bounding the "volume" of the intersection of an $O(\sqrt{n})$ neighborhood of $\sum_{j=1}^n \pi_j$ and $\mathcal{H}_B$. We prove that, in d_0 dimensions, the volume is large, and each such dimension contributes a multiplicative $O(1)$ factor to the likelihood; in d_Δ dimensions, the volume is $O(B)$, and each such dimension contributes a multiplicative $O\left(\frac{B+1}{\sqrt{n}}\right)$ factor to the likelihood; and in the remaining $q - d_\infty$ dimensions, the volume is $O(1)$, and each such dimension contributes a multiplicative $O(\frac{1}{\sqrt{n}})$ factor to the likelihood. Putting all together, this proves the desired upper bound

$$O(1)^{d_0} \times O\left(\frac{B+1}{\sqrt{n}}\right)^{d_\Delta} \times O\left(\frac{1}{\sqrt{n}}\right)^{q-d_\infty} = O\left(\frac{(B+1)^{d_\Delta}}{(\sqrt{n})^{q-d_0}}\right)$$

To prove the polynomial lower bound, we first pretend that the PMV can take non-integer values, then enumerate (possibly non-integral) vectors that are far away from each other by exploring two directions. The first direction is the convex hull of $\mathcal{C}_0$, which is a $(d_0 - 1)$-dimensional space that represents no budget ($B = 0$), and each such dimension contributes an $\Omega(\sqrt{n})$ multiplicative factor to the total number of desirable vectors. The second direction is the convex hull of $\mathcal{C}_\infty$, which is a d_Δ-dimensional space that represents infinite budget ($B = \infty$), and each such dimension contributes an $\Omega(B + 1)$ multiplicative factor to the total number of desirable vectors. Then, we prove that for each such (possibly non-integral) vector, there exists a nearby integer vector, and apply the pointwise concentration bound [77, Lemma 1] to prove the desired lower bound. The full proof can be found in Appendix E.2.

C.4 Step 3 of The Proof: Prove and Apply the PMV-Multi-Instability Theorem

In this subsection, we first extend Theorem 2 to solve the PMV-multi-instability problem (Definition 6) in Theorem 3, then apply it to prove Theorem 1. For every $i \le I$, we use superscript i to denote the notation defined for $\mathcal{S}^i$. For example, d_0^i, and d_Δ^i denote d_0 and d_Δ for $\mathcal{S}^i$. To present the result, it is convenient to define the following graph.

Definition 12 (**Activation graph for PMV-multi-instability**). *Given a PMV multi-instability setting $\mathcal{M} = \{\mathcal{S}^i : i \le I\}$, n, and B, we define a weighted undirected bipartite graph, called* activation graph *and is denoted by $\mathcal{A}_{n,B}$, as follows.*

- **Vertices.** *There are two sides of vertices: $CH(\Pi)$ and $\{\mathcal{S}^1, \dots, \mathcal{S}^I\}$.*
- **Edges and weights.** *For any $\pi \in CH(\Pi)$ and any $\mathcal{S}^i \in \mathcal{M}$, the weight on the edge between π and $\mathcal{S}^i$, denoted by $w_{n,B}(\pi, \mathcal{S}^i)$, is defined as follows: for every PMV-instability setting $\mathcal{S}$, define*

$$w_{n,B}(\pi, \mathcal{S}) \triangleq \begin{cases} -\infty & \text{if } \mathcal{U}_{n,B} = \emptyset \\ -\frac{2n}{\log n} & \text{if } \mathcal{U}_{n,B} \neq \emptyset \text{ and } \pi \notin \mathcal{C}_0 \\ d_0 + d_\Delta \cdot \min\left\{\frac{2\log(B+1)}{\log n}, 1\right\} & \text{otherwise} \end{cases} \tag{8}$$

Notice that while the conditions in (8) depend on π, the values of $w_{n,B}(\pi, \mathcal{S})$ do not depend on π, and they are chosen so that $(\sqrt{n})^{w_{n,B}}$ corresponds to the values in the exponential cases and polynomial cases of Theorem 2. Specifically, the $-\frac{2n}{\log n}$ value in the second case is chosen so that $(\sqrt{n})^{-\frac{2n}{\log n}} = \exp(-n)$, which corresponds to the exponential cases.

Given the PMV multi-instability setting $\mathcal{M} = \{\mathcal{S}^i : i \le I\}$, B, and n, let $w_{\max}$ denote the maximum weigh on edges in $\mathcal{A}_{n,B}$ and let $(\pi_{\max}, i_{\max})$ denote an arbitrary edge with the max weight. That is,

$$w_{\max} \triangleq \max_{\pi \in \text{CH}(\Pi), i \le I} \left\{w_{n,B}(\pi, \mathcal{S}^i)\right\} \text{ and } (\pi_{\max}, i_{\max}) \triangleq \arg\max_{\pi \in \text{CH}(\Pi), i \le I} w_{n,B}(\pi, \mathcal{S}^i)$$

Let $w_{\min}$ denote the weight of the minimax weighted edge in $\mathcal{A}_{n,B}$, denoted by $(\pi_{\min}, i_{\min})$, where min is taken over all $\pi \in \text{CH}(\Pi)$ and max is taken over all edges connected to π. That is,

$$w_{\min} \triangleq \min_{\pi \in \text{CH}(\Pi)} \max_{i \le I} \left\{w_{n,B}(\pi, \mathcal{S}^i)\right\}, \pi_{\min} \triangleq \arg\min_{\pi \in \text{CH}(\Pi)} \max_{i \le I} w_{n,B}(\pi, \mathcal{S}^i),$$
$$\text{and } i_{\min} \triangleq \arg\max_{i \le I} w_{n,B}(\pi_{\min}, \mathcal{S}^i)$$

Notice that $i_{\max}$ and $i_{\min}$ are both in $[q]$ and $w_{\max}$, $w_{\min}$, $i_{\max}$, and $i_{\min}$ depend on B and n, which are clear from the context.

Theorem 3 (**Semi-Random PMV-multi-instability, $B = O(n)$**). *Given any $q \in \mathbb{N}$, any closed and strictly positive Π over $[q]$, and any PMV multi-instability setting $\mathcal{M} = \{\mathcal{S}^i : i \le I\}$, there exists a constant $C_1 > 0$ so that for any $n \in \mathbb{N}$ and any $B \ge 0$ with $B \le C_1 n$,*

$$\sup_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}^{\mathcal{M}}_{n,B}\right) = \begin{cases} 0 & \text{if } w_{\max} = -\infty \\ \exp(-\Theta(n)) & \text{if } w_{\max} = -\frac{2n}{\log n} \\ \Theta\left(\left(\frac{1}{\sqrt{n}}\right)^{q - w_{\max}}\right) & \text{otherwise} \end{cases}$$

$$\inf_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}^{\mathcal{M}}_{n,B}\right) = \begin{cases} 0 & \text{if } w_{\min} = -\infty \\ \exp(-\Theta(n)) & \text{if } w_{\min} = -\frac{2n}{\log n} \\ \Theta\left(\left(\frac{1}{\sqrt{n}}\right)^{q - w_{\min}}\right) & \text{otherwise} \end{cases}$$

The proof can be found in Appendix F.1. Notice that in Theorem 3, when $w_{\max} > 0$ and $w_{\min} > 0$, we have

$$\left(\frac{1}{\sqrt{n}}\right)^{q - w_{\max}} = \frac{\min\{B+1, \sqrt{n}\}^{d_{\Delta}^{i_{\max}}}}{(\sqrt{n})^{q - d_0^{i_{\max}}}} \text{ and } \left(\frac{1}{\sqrt{n}}\right)^{q - w_{\min}} = \frac{\min\{B+1, \sqrt{n}\}^{d_{\Delta}^{i_{\min}}}}{(\sqrt{n})^{q - d_0^{i_{\min}}}} \tag{9}$$

Recall from Lemma 1 that any coalitional influence problem $X \in \{\text{CM}, \text{MoV}\}$ under any GSR r can be represented by a PMV-multi-instability problem. Therefore, Theorem 1 follows immediately after Theorem 3 and (9). □

D Materials for Section 3.1

D.1 Full Version of Lemma 1 and Its Proof

Lemma 1. (Coalitional Influence as PMV-multi-instability, Full Version). *For any coalitional influence problem $X \in \{\text{CM}, \text{MoV}, \text{CB}_{d,\vec{c}}, \text{DB}_{d,\vec{c}}, \text{E-CB}_{d,\vec{c}}, \text{E-}$ and any GSR r, there exist a set $\mathcal{M} = \{\mathcal{S}^i : i \le I\}$ of I PMV-instability settings such that for every n-profile P and every $B \ge 0$, $X(r, P, B) = 1$ if and only if $Hist(P) \in \mathcal{U}^{\mathcal{M}}_{n,B}$.*

Proof. We first recall some formal notation about GSR. For any real number x, let $\text{Sign}(x) \in \{+, -, 0\}$ denote the sign of x. Given a set of K hyperplanes in the q-dimensional Euclidean space, denoted by $\vec{H} = (\vec{h}_1, \ldots, \vec{h}_K)$, for any $\vec{x} \in \mathbb{R}^q$, we let $\text{Sign}_{\vec{H}}(\vec{x}) = (\text{Sign}(\vec{x} \cdot \vec{h}_1), \ldots, \text{Sign}(\vec{x} \cdot \vec{h}_K))$. In other words, for any $k \le K$, the k-th component of $\text{Sign}_{\vec{H}}(\vec{x})$ equals to 0, if $\vec{p}$ lies in hyperplane $\vec{h}_k$; and it equals to + (respectively, −) if $\vec{p}$ lies in the positive (respectively, negative) side of $\vec{h}_k$. Each element in $\{+, -, 0\}^K$ is called a *signature.*

Definition 13 (**Feasible signatures**). *Given integer $\vec{H}$ with $K = |\vec{H}|$, let $\mathcal{S}_K = \{+, -, 0\}^K$. A signature $\vec{t} \in \mathcal{S}_K$ is* feasible, *if there exists $\vec{x} \in \mathbb{R}^{m!}$ such that $\text{Sign}_{\vec{H}}(\vec{x}) = \vec{t}$. Let $\mathcal{S}_{\vec{H}} \subseteq \mathcal{S}_K$ denote the set of all feasible signatures.*

The domain of any GISR $\overline{r}$ can be naturally extended to $\mathbb{R}^{m!}$ and to $\mathcal{S}_{\vec{H}}$. Specifically, for any $\vec{t} \in \mathcal{S}_{\vec{H}}$ we let $\overline{r}(\vec{t}) = g(\vec{t})$. It suffices to define g on the feasible signatures, i.e., $\mathcal{S}_{\vec{H}}$.

See [78, Section D.2] for the GSR representations of some commonly-studied voting rules, especially the rules defined in Appendix B.1.

Next, given $\vec{H}$ and a feasible signature $\vec{t}$, we recall from [78, Section D.4] the definition of $\mathcal{H}^{\vec{H},\vec{t}}$ that represents profiles whose signatures are $\vec{t}$.

Definition 14 ($\mathcal{H}^{\vec{H},\vec{t}}$ (**$\mathcal{H}^{\vec{t}}$ in short**)). *For any $\vec{H} = (\vec{h}_1, \ldots, \vec{h}_K) \in (\mathbb{R}^d)^K$ and any $\vec{t} \in \mathcal{S}_{\vec{H}}$, we let $\mathbf{A}^{\vec{t}} = \begin{bmatrix} \mathbf{A}^{\vec{t}}_+ \\ \mathbf{A}^{\vec{t}}_- \\ \mathbf{A}^{\vec{t}}_0 \end{bmatrix}$, where*

- $\mathbf{A}^{\vec{t}}_+$ *consists of a row $-\vec{h}_i$ for each $i \le K$ with $t_i = +$.*
- $\mathbf{A}^{\vec{t}}_-$ *consists of a row $\vec{h}_i$ for each $i \le K$ with $t_i = -$.*
- $\mathbf{A}^{\vec{t}}_0$ *consists of two rows $-\vec{h}_i$ and $\vec{h}_i$ for each $i \le K$ with $t_i = 0$.*

Let $\mathbf{b}^{\vec{t}} = [\underbrace{-\vec{1}}_{\text{for } \mathbf{A}^{\vec{t}}_+}, \underbrace{-\vec{1}}_{\text{for } \mathbf{A}^{\vec{t}}_-}, \underbrace{\vec{0}}_{\text{for } \mathbf{A}^{\vec{t}}_0}]$. *The corresponding polyhedron is denoted by $\mathcal{H}^{\vec{H},\vec{t}}$, or $\mathcal{H}^{\vec{t}}$ in short when $\vec{H}$ is clear from the context.*

Then, we formally define some vote operations that will be used in the proof.

Definition 15. *We define four vote operations as follows.*

- **Vote change**: *$O_{\pm} = \{Hist(R_2) - Hist(R_1) : R_1, R_2 \in \mathcal{L}(\mathcal{A})\}$.*
- **Motivated vote change**: *for any pair of different alternatives a, b, let*

$$\mathbb{O}^{a \to b}_{\pm} = \{Hist(R_b) - Hist(R_a) : R_b, R_a \in \mathcal{L}(\mathcal{A}) \text{ and } b \succ_{R_a} a\}$$

- **Generalized vote change**: *for any pair of different alternatives a, b, let*

$$\mathbb{O}^*_\pm = \{Hist(R_1), Hist(R_2), Hist(R_2) - Hist(R_1) : R_1, R_2 \in \mathcal{L}(\mathcal{A})\}$$

We are now ready to define the PMV-instability settings whose union models the coalitional influence problems described in the statement of the proposition.

- $X =$ CM. For every pair of different alternatives a, b, and every pair of feasible signatures $\vec{t}_a, \vec{t}_b$ such that $r(\vec{t}_a) = \{a\}$ and $r(\vec{t}_b) = \{b\}$, $\mathcal{M}$ contains
$$\langle \mathcal{H}^{\vec{t}_a}, \mathcal{H}^{\vec{t}_b}, \mathbb{O}^{a\to b}_\pm, \vec{1}\,\rangle$$
- $X =$ MoV. For every pair of different alternatives a, b, and every pair of feasible signatures $\vec{t}_a, \vec{t}_b$ such that $r(\vec{t}_a) = \{a\}$ and $r(\vec{t}_b) = \{b\}$, $\mathcal{M}$ contains
$$\langle \mathcal{H}^{\vec{t}_a}, \mathcal{H}^{\vec{t}_b}, \mathbb{O}_\pm, \vec{1}\,\rangle$$
- $X =$ CB$_{d,\vec{c}}$. For every every pair of feasible signatures $\vec{t}, \vec{t}_a$ such that $r(\vec{t}_a) = \{a\}$, $\mathcal{M}$ contains
$$\langle \mathcal{H}^{\vec{t}}, \mathcal{H}^{\vec{t}_a}, \mathbb{O}^*_\pm, \vec{c}\,\rangle$$
- $X =$ DB$_{d,\vec{c}}$. For every alternative $b \neq a$, and every pair of feasible signatures $\vec{t}, \vec{t}_b$ such that $r(\vec{t}_b) = \{b\}$, $\mathcal{M}$ contains
$$\langle \mathcal{H}^{\vec{t}}, \mathcal{H}^{\vec{t}_b}, \mathbb{O}^*_\pm, \vec{c}\,\rangle$$
- $X =$ E-CB$_{d,\vec{c}}$. For every every pair of feasible signatures $\vec{t}, \vec{t}_a$ such that $r(\vec{t}) \neq \{a\}$ and $r(\vec{t}_a) = \{a\}$, $\mathcal{M}$ contains
$$\langle \mathcal{H}^{\vec{t}}, \mathcal{H}^{\vec{t}_a}, \mathbb{O}^*_\pm, \vec{c}\,\rangle$$
- $X =$ E-DB$_{d,\vec{c}}$. For every alternative $b \neq a$, and every pair of feasible signatures $\vec{t}_a, \vec{t}_b$ such that $r(\vec{t}_a) = \{a\}$ and $r(\vec{t}_b) = \{b\}$, $\mathcal{M}$ contains
$$\langle \mathcal{H}^{\vec{t}_a}, \mathcal{H}^{\vec{t}_b}, \mathbb{O}^*_\pm, \vec{c}\,\rangle$$

□

E Materials for Section C.3

E.1 Properties of B_{Π^*} and $B^-_{\Pi^*}$

Claim 1. *For any convex and compact set Π^*, if $B_{\Pi^*} \neq \infty$ then $\Pi^* \cap C_{B_{\Pi^*}} \neq \emptyset$.*

Proof. Let $\{B_j : j \in \mathbb{N}\}$ denote a sequence that converges to B_{Π^*}, such that for all $j \in \mathbb{N}$, $\Pi^* \cap C_{B_j} \neq \emptyset$. For any $j \in \mathbb{N}$, let $\vec{y}_j \in \Pi^* \cap C_{B_j}$ denote an arbitrary vector. Because Π^* is compact, a subsequence of $\{\vec{y}_j : j \in \mathbb{N}\}$, denoted by $\{\vec{y}_{j_i} : i \in \mathbb{N}\}$ converges to a vector $\vec{y}^* \in \Pi^*$. Notice that $\vec{y}_{j_i}$ is in Π^*, $\mathcal{H}_{\mathrm{S},\leq 0}$, and both are closed sets. Therefore, $\vec{y}^* \in \Pi^* \cap \mathcal{H}_{\mathrm{S},\leq 0}$.

Next, we prove that $\vec{y}^* \in C_{B_{\Pi^*}}$. For every $i \in \mathbb{N}$, let $\vec{o}_{j_i} \in \mathbb{R}^{|\mathbb{O}|}_{\geq 0}$ denote the operation vector such that $\vec{c} \cdot \vec{o}_{j_i} \leq B_{j_i}$ and $\vec{y}_{j_i} + \vec{o}_{j_i} \times \mathbf{O} \in \mathcal{H}_{\mathrm{T},\leq 0}$. Let $\vec{x}_{j_i} = \vec{y}_{j_i} + \vec{o}_{j_i} \times \mathbf{O}$. Because $\vec{o}_{j_i}$'s are bounded ($\vec{c} \cdot \vec{o}_{j_i} \leq B_1$), there exists a subsequence $\{j'_1 : i \in \mathbb{N}\}$ of $\{j_1 : i \in \mathbb{N}\}$ such that $\vec{o}_{j'_i}$ converges to a vector $\vec{o}^*$. It is not hard to verify that $\vec{c} \cdot \vec{o}^* \leq B_{\Pi^*}$ and $\{\vec{x}_{j'_i} = \vec{y}_{j'_i} + \vec{o}_{j'_i} \times \mathbf{O} : i \in \mathbb{N}\}$ converges to $\vec{y}^* + \vec{o}^* \times \mathbf{O}$. Because for all $i \in \mathbb{N}$, $\vec{x}_{j'_i} \in \mathcal{H}_{\mathrm{T},\leq 0}$ and $\mathcal{H}_{\mathrm{T},\leq 0}$ is closed, we have $\vec{y}^* + \vec{o}^* \times \mathbf{O} \in \mathcal{H}_{\mathrm{T},\leq 0}$ as well. This proves that $\vec{y}^* \in C_{B_{\Pi^*}}$, which completes the proof of Claim 1. □

Claim 2. *For any bounded set Π^*, if $\Pi^* \subseteq C_\infty$ then $B^-_{\Pi^*} \neq \infty$.*

Proof. It suffices to prove that there exists $B^* \geq 0$ so that $\Pi^* \subseteq C_{B^*}$. Because Π^* is bounded, let Q denote any cube that contains Π^*. Because Q is a polytope and C_∞ is a polyhedral cone, $Q \cap C_\infty$ is a polytope that contains Π^*. Let the V-representation of $Q \cap C_\infty$ be $\mathrm{CH}(\{\vec{x}_1, \ldots, \vec{x}_k\})$ for some $k \in \mathbb{N}$. For every $j \leq k$, because $\vec{x} + j \in C_\infty$, there exists $B_j \geq 0$ such that $\vec{x}_j \in C_{B_j}$. Let $B^* = \max\{B_1, \ldots, B_k\}$. It follows that $\Pi^* \subseteq Q \cap C_\infty \subseteq C_{B^*}$, which proves Claim 2. □

E.2 Proof of Theorem 2

Theorem 2. (Semi-Random Likelihood of PMV-Instability Problem). *Given any $q \in \mathbb{N}$, any closed and strictly positive Π over $[q]$, and any PMV-instability setting $\langle \mathcal{H}_S, \mathcal{H}_T, \mathbb{O}, \vec{c}\,\rangle$, any $C_2 > 0$ and $C_3 > 0$ with $C_2 < B_{CH(\Pi)} < C_3$, any $n \in \mathbb{N}$, and any $B \geq 0$,*

$\sup_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}\right) =$

Name	Likelihood		Condition
0 case	0		κ_1
exp case	$\exp(-\Theta(n))$		$\neg\kappa_1 \wedge \kappa_2$
PT-$\Theta(\sqrt{n})$	$\Theta\left(\dfrac{\min\{B+1,\sqrt{n}\}^{d_\Delta}}{(\sqrt{n})^{q-d_0}}\right)$		$\neg\kappa_1 \wedge \neg\kappa_3$
PT-$\Theta(n)$	$\exp(-\Theta(n))$	*if* $B \leq C_2 n$	*otherwise, i.e.,* $\neg\kappa_1 \wedge \neg\kappa_2 \wedge \kappa_3$
	$\Theta\left((\frac{1}{\sqrt{n}})^{q-d_\infty}\right)$	*if* $B \geq C_3 n$	

For any $C_2^- < B^-_{CH(\Pi)} < C_3^-$, any $n \in \mathbb{N}$, and any $B \geq 0$,

$$\inf_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}\right) = \left\{\begin{array}{|l|l|l|l|}\hline \textbf{Name} & \textbf{Likelihood} & & \textbf{Condition} \\ \hline \textit{0 case} & 0 & & \kappa_1 \\ \hline \textit{exp case} & \exp(-\Theta(n)) & & \neg\kappa_1 \wedge \kappa_4 \\ \hline \textit{PT-}\Theta(\sqrt{n}) & \Theta\left(\dfrac{\min\{B+1,\sqrt{n}\}^{d_\Delta}}{(\sqrt{n})^{q-d_0}}\right) & & \neg\kappa_1 \wedge \neg\kappa_5 \\ \hline \textit{PT-}\Theta(n) & \exp(-\Theta(n)) & \textit{if } B \leq C_2^- n & \textit{otherwise, i.e.,} \\ & \Theta\left((\frac{1}{\sqrt{n}})^{q-d_\infty}\right) & \textit{if } B \geq C_3^- n & \neg\kappa_1 \wedge \neg\kappa_4 \wedge \kappa_5 \\ \hline \end{array}\right.$$

Proof. We first prove the sup part of the theorem, then leverage the techniques to prove the inf part.

Proof for the sup part. For convenience, hyperlinks (in red) to the proofs of four cases are provided as follows.

$$\sup_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}\right) = \left\{\begin{array}{|l|c|c|c|c|}\hline \text{0 case} & 0 & \kappa_1 & & \\ \hline \text{exponential case} & \exp(-\Theta(n)) & \kappa_2 & & \\ \hline \text{PT-}\Theta(n)\text{-sup} & \exp(-\Theta(n)) & B \leq C_2 n & \neg\kappa_2 \wedge \kappa_3 & \neg\kappa_1 \\ & \Theta\left((\dfrac{1}{\sqrt{n}})^{q-d_\infty}\right) & B \geq C_3 n & \neg\kappa_2 \wedge \kappa_3 & \\ \hline \text{PT-}\Theta(\sqrt{n})\text{-sup} & \Theta\left(\dfrac{\min\{B,\sqrt{n}\}^{d_\Delta}}{(\sqrt{n})^{q-d_0}}\right) & \neg\kappa_3 & & \\ \hline \end{array}\right.$$

Proof for the 0 case of sup is straightforward, because $\mathcal{U}_{n,B} = \emptyset$ states that the PMV-instability problem does not have a size-n non-negative integer solution. In the rest of the proof for sup, it suffices prove the exponential case and the polynomial case for all n that are larger than a constant N. This is because for any n such that the 0 case does not hold (which means that $\mathcal{U}_{n,B} \neq \emptyset$), and for every $\vec{\pi} \in \Pi^n$,

$$\Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}\right) \in [\epsilon^n, 1]$$

Therefore, for every n below a constant N, we have

$$\sup_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}\right) \in [\epsilon^N, 1],$$

Proof for the exponential case of sup. Because $\text{CH}(\Pi) \cap \mathcal{C}_\infty = \emptyset$, $\text{CH}(\Pi)$ is convex and compact, and $\mathcal{C}_\infty$ is convex, due to the strict hyperplane separation theorem, for every $\vec{\pi} \in \Pi^n$, $\sum_{i=1}^n \pi_i$ is $\Omega(n)$ away from any vector in $\mathcal{C}_\infty$, which means that $\sum_{i=1}^n \pi_i$ is $\Omega(n)$ away from any vector in $\mathcal{H}_\infty$, because $\mathcal{C}_\infty$ is the characteristic cone of $\mathcal{H}_\infty$ as proved in the following claim.

Claim 3. *The characteristic cone of $\mathcal{H}_\infty$ is $\mathcal{C}_\infty$.*

Proof. We prove two general observations about characteristic cones. For each $i \in \{1,2\}$, let $\mathcal{H}^i$ denote a polyhedron whose V-representation is $\mathcal{V}_i + \mathcal{H}^i_{\leq 0}$, where $\mathcal{V}_i$ is a convex polytope and $\mathcal{H}^i_{\leq 0}$ is the characteristic cone of $\mathcal{H}^i$.

Observation 1. The characteristic cone of $\mathcal{H}^1 + \mathcal{H}^2$ is $\mathcal{H}^1_{\leq 0} + \mathcal{H}^2_{\leq 0}$. This is because

$$\mathcal{H}^1 + \mathcal{H}^2 = (\mathcal{V}_1 + \mathcal{V}_2) + (\mathcal{H}^1_{\leq 0} + \mathcal{H}^2_{\leq 0})$$

Here $\mathcal{H}^1_{\leq 0} + \mathcal{H}^2_{\leq 0}$ is indeed a finitely generated cone, because suppose for $i \in \{1,2\}$, $\mathcal{H}^i_{\leq 0}$ is the convex cone generated from $\mathcal{B}_i$, then it is not hard to verify that $\mathcal{H}^1_{\leq 0} + \mathcal{H}^2_{\leq 0}$ is a cone generated by $\mathcal{B}_1 \cup \mathcal{B}_2$.

Observation 2. If $\mathcal{H}^1 \cap \mathcal{H}^2 \neq \emptyset$, then its characteristic cone is $\mathcal{H}^1_{\leq 0} \cap \mathcal{H}^2_{\leq 0}$. This is proved by the H-representations of $\mathcal{H}^1$ and $\mathcal{H}^2$. Suppose for each $i \in \{1,2\}$, $\mathcal{H}^i = \{\vec{x} : \mathbf{A}_i \times (\vec{x})^\top \leq (\mathbf{b}_i)^\top\}$, which means that $\mathcal{H}^i_{\leq 0} = \left\{\vec{x} : \mathbf{A}_i \times (\vec{x})^\top \leq \left(\vec{0}\right)^\top\right\}$. Then, we have

$$\mathcal{H}^1 \cap \mathcal{H}^2 = \left\{\vec{x} : \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \times (\vec{x})^\top \leq (\mathbf{b}_1, \mathbf{b}_2)^\top\right\},$$

whose characteristic cone is $\left\{\vec{x} : \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \times (\vec{x})^\top \leq \left(\vec{0}\right)^\top\right\} = \mathcal{H}^1_{\leq 0} \cap \mathcal{H}^2_{\leq 0}$.

Recall that $\mathcal{H}_\infty = \mathcal{H}_{\text{S}} \cap (\mathcal{H}_{\text{T}} + \mathcal{Q}_\infty)$. By Observation 1, the characteristic cone of $\mathcal{H}_{\text{T}} + \mathcal{Q}_\infty$ is $\mathcal{H}_{\text{T},\leq 0} + \mathcal{Q}_\infty$. Then, by Observation 2, the characteristic cone of $\mathcal{H}_\infty$ is $\mathcal{H}_{\text{S},\leq 0} \cap \left(\mathcal{H}_{\text{T},\leq 0} + \mathcal{Q}_\infty\right)$, which is $\mathcal{C}_\infty$ (due to (5)). □

Then, the upper bound in the exponential case of sup follows after a straightforward application of Hoeffding's inequality and the union bound (applied to all q dimension). More precisely, recall that $\vec{\pi}$ is strictly positive, then Hoeffding's inequality implies that for every $i \in [q]$, the probability for the ith dimension of $\vec{X}_{\vec{\pi}}$ to be more than $\Omega(n)$ away from the ith dimension of $\sum_{j=1} \pi_j$ is exponentially small. Therefore, according to the union bound, the probability for the L_∞ distance between $\vec{X}_{\vec{\pi}}$ and $\sum_{j=1} \pi_j$ to be $\Omega(n)$ is exponentially small, which implies that the probability for $\vec{X}_{\vec{\pi}}$ to be in $\mathcal{U}_{n,B}$ is exponentially small. The lower bound in the exponential case is straightforward, because for any $\vec{\pi} \in \Pi^n$ (recall that all distributions in $\vec{\pi}$ are strictly positive) and any $\vec{x} \in \mathcal{U}_{n,B}$, $\Pr(\vec{X}_{\vec{\pi}} = \vec{x}) = \exp(-\Theta(n))$.

Proof for the phase transition at $\Theta(\sqrt{n})$ case of sup (PT-$\Theta(\sqrt{n})$-sup for short). The proof proceeds in the following three steps:

- Prove the polynomial upper bound for $B \le \sqrt{n}$, i.e., $O\left(\frac{(B+1)^{d_\Delta}}{(\sqrt{n})^{q-d_0}}\right)$.
- Prove the polynomial upper bound for $B > \sqrt{n}$, i.e., $O\left((\frac{1}{\sqrt{n}})^{q-d_\infty}\right)$.
- Prove the asymptotically matching polynomial lower bound, i.e., $\Omega\left(\frac{\min\{B+1, \sqrt{n}\}^{d_\Delta}}{(\sqrt{n})^{q-d_0}}\right)$

We first introduce some notation and assumptions that will be used in the proofs. Let $\mathbf{A}_0 = \begin{bmatrix} \mathbf{A}_\mathrm{S} \\ \mathbf{A}_\mathrm{T} \end{bmatrix}$ and let $\mathbf{A}_\infty$ denote an integer matrix that characterizes $\mathcal{C}_\infty$. That is,

$$\mathcal{C}_\infty = \left\{\vec{x} \in \mathbb{R}^q : \mathbf{A}_\infty \times (\vec{x})^\top \le \left(\vec{0}\right)^\top\right\}$$

The existence of such $\mathbf{A}_\infty$ is due to [13, Proposition 3.12], which states that any polyhedron that has a rational H-representation has a rational V-representation, and vice versa. More precisely, because $\mathcal{H}_{\mathrm{T},\le 0}$ has a rational H-representation, it has a rational V-representation, denoted by $\mathrm{Cone}(\{\vec{x}_1, \ldots, \vec{x}_k\})$, where $\{\vec{x}_1, \ldots, \vec{x}_k\} \subseteq \mathbb{Q}^q$. Then, we have

$$\mathcal{H}_{\mathrm{T},\le 0} + \mathcal{Q}_\infty = \mathrm{Cone}(\{\vec{x}_1, \ldots, \vec{x}_k\} \cup \mathbb{O}),$$

which means that $\mathcal{H}_{\mathrm{T},\le 0} + \mathcal{Q}_\infty$ can be represented by a set of linear inequalities with rational coefficients, due to [13, Proposition 3.12]. Consequently, $\mathcal{H}_{\mathrm{S},\le 0} \cap (\mathcal{H}_{\mathrm{T},\le 0} + \mathcal{Q}_\infty)$ can be represented by a set of linear inequalities with rational coefficients, by combining the linear inequalities for $\mathcal{H}_{\mathrm{S},\le 0}$ and the linear inequalities for $\mathcal{H}_{\mathrm{T},\le 0} + \mathcal{Q}_\infty$.

Let $\mathbf{A}_0^=$ and $\mathbf{A}_\infty^=$ denote the implicit equalities of $\mathbf{A}_0$ and $\mathbf{A}_\infty$, respectively. We have $\mathrm{Rank}(\mathbf{A}_0^=) = q - d_0$ and $\mathrm{Rank}(\mathbf{A}_\infty^=) = q - d_\infty$ ([13, Theorem 3.17]). Next, we show that, without loss of generality, in the rest of the proof for PT-$\Theta(\sqrt{n})$-sup, we can assume that $\vec{1}$ cannot be represented as a linear combination of rows in $\mathbf{A}_0^=$ or a linear combination of rows in $\mathbf{A}_\infty^=$. Formally,

Assumption 1. *Rank* $\left(\begin{bmatrix} \mathbf{A}_0^= \\ \vec{1} \end{bmatrix}\right) = q - d_0 + 1$.

Assumption 2. *Rank* $\left(\begin{bmatrix} \mathbf{A}_\infty^= \\ \vec{1} \end{bmatrix}\right) = q - d_\infty + 1$.

To see that we can assume Assumption 1, suppose for the sake of contradiction that Assumption 1 does not hold, which means that $\vec{1}$ is a linear combination of rows in $\mathbf{A}_0^=$. Then, for every $\vec{x} \in \mathcal{C}_0 = \mathcal{H}_{\mathrm{S},\le 0} \cap \mathcal{H}_{\mathrm{T},\le 0}$, we have $\vec{x} \cdot 1 = 0$. Therefore, $\mathrm{CH}(\Pi) \cap \mathcal{C}_0 = \emptyset$, which contradicts $\neg\kappa_3$. Similarly, if Assumption 2 does not hold, then we have $\mathrm{CH}(\Pi) \cap \mathcal{C}_\infty = \emptyset$, which again contradicts $\neg\kappa_3$, because $\mathcal{C}_0 \subseteq \mathcal{C}_\infty$.

Proof for the polynomial upper bound of PT-$\Theta(\sqrt{n})$-sup, $B \le \sqrt{n}$.

Overview of proof. The proof proceeds in three steps. In Step 1, we use $\mathbf{A}_0^=$ and $\mathbf{A}_\infty^=$ to define a partition of $[q]$ into three sets I_{0+}, I_{0-}, and I_1, which contain $q - d_0 + 1$, $d_\infty - d_0$, and $d_\infty - 1$ numbers, respectively. For convenience, we rename the coordinates so that

$$\underbrace{1, \ldots, q - d_\infty + 1}_{I_{0+}}, \underbrace{q - d_\infty + 2, \ldots, q - d_0 + 1}_{I_{0-}}, \underbrace{q - d_0 + 2, \ldots, q}_{I_1}$$

Let $I_{1+} = I_{0-} \cup I_1$ and let $I_0 = I_{0+} \cup I_{0-}$. Step 2 proves two properties of the partition. Let $\mathcal{H}_{B,n} = \{\vec{x} \in \mathcal{H}_B : \vec{x} \cdot \vec{1} = n\}$ denote the subset of $\mathcal{H}_B$ that consists of all size-n vectors. First, in Step 2.1, we prove that given the I_{1+} coordinates of vectors in $\mathcal{H}_{B,n}$, each of its remaining coordinates (in I_{0+}) can take no more than $O(1)$ integer values. Second, in Step 2.2, we prove that given the I_1 coordinates of vectors in $\mathcal{H}_{B,n}$, each of its remaining coordinates (in I_0) can take no more than $O(B)$ integer values.

In light of Step 2.1 and Step 2.2, we can enumerate integer vectors $\vec{y}$ in $\mathcal{H}_{B,n}$ as follows: first, we fix the I_1 coordinates of $\vec{y}$; second, each of the $d_\Delta = d_\infty - d_0$ coordinates in I_{0-} takes no more than $O(B)$ integer values; and finally, each of the $q - d_\infty + 1$ coordinates in I_{0+} takes no more than $O(1)$ integer values. Then in Step 3, we leverage this enumeration method with the Bayesian network representation and the point-wise anti-concentration bound in [77] to prove the upper bound.

Step 1 of poly upper bound: Define the partition $[q] = I_{0+} \cup I_{0-} \cup I_1$. Let $\mathcal{P}_0$ and $\mathcal{P}_\infty$ denote the affine hulls of C_0 and C_∞, respectively (which are the same as the linear spaces generated by C_0 and C_∞, because both contains $\vec{0}$). It follows from [13, Theorem 3.17] that

$$\mathcal{P}_0 = \left\{\vec{x} \in \mathbb{R}^q : \mathbf{A}_0^= \times (\vec{x})^\top = \left(\vec{0}\right)^\top\right\} \text{ and } \mathcal{P}_\infty = \left\{\vec{x} \in \mathbb{R}^q : \mathbf{A}_\infty^= \times (\vec{x})^\top = \left(\vec{0}\right)^\top\right\} \tag{10}$$

That is, $\mathcal{P}_0$ (respectively, $\mathcal{P}_\infty$) consists of vectors that satisfies of the implicit equalities of $\mathbf{A}_0$ (respectively, $\mathbf{A}_\infty$).

Let $\mathbf{A}_* = \begin{bmatrix} \mathbf{A}_\infty^= \\ \mathbf{A}_0^= \end{bmatrix}$. We will define a partition of $[q]$ as $I_{0+} \cup I_{0-} \cup I_1$, where $|I_{0+}| = q - d_\infty + 1$, $|I_{0-}| = d_\Delta$, and $|I_1| = d_0 - 1$, and the partition satisfies the following two conditions.

- **Condition 1.** The I_{0+} columns of $\begin{bmatrix} \mathbf{A}_\infty^= \\ \vec{1} \end{bmatrix}$ are linearly independent.
- **Condition 2.** The $I_{0+} \cup I_{0-}$ columns of $\begin{bmatrix} \mathbf{A}_0^= \\ \vec{1} \end{bmatrix}$ are linearly independent.

We first define two sets I'_{0+} and I'_{0-} as follows.

Define I'_{0+}. Recall that $\text{Rank}(\mathbf{A}_\infty^=) = q - d_\infty$. Therefore, $\mathbf{A}_\infty^=$ contains a set of $q - d_\infty$ linearly independent column vectors, whose indices are denoted by $I'_{0+} \subseteq [q]$. W.l.o.g. let $I'_{0+} = \{1, \ldots, q - d_\infty\}$—if this is not the case, then we shift the I'_{0+} columns in $\mathbf{A}_\infty^=$ to be the first $q - d_\infty$ columns and rename the coordinates.

Define I'_{0-}. Notice that the I'_{0+} columns of $\mathbf{A}_*$ are linearly independent (because their $\mathbf{A}_\infty^=$ parts are already linearly independent). Because $C_0 \subseteq C_\infty$, we have $\mathcal{P}_0 \subseteq \mathcal{P}_\infty$, which means that

$$\left\{\vec{x} \in \mathbb{R}^q : \mathbf{A}_* \times (\vec{x})^\top = \left(\vec{0}\right)^\top\right\} = \mathcal{P}_\infty \cap \mathcal{P}_0 = \mathcal{P}_0$$

This means that $\text{Rank}(\mathbf{A}_*) = \text{Rank}(\mathbf{A}_0^=) = q - d_0$. Consequently, there exist a set of $q - d_0 - (q - d_\infty) = d_\infty - d_0 = d_\Delta$ columns of $\mathbf{A}_*$, whose indices are denoted by $I'_{0-} \subseteq ([q] \setminus I'_{0+})$, such the $I'_{0+} \cup I'_{0-}$ columns of $\mathbf{A}_*$ are linearly independent. W.l.o.g., let $I'_{0-} = \{q - d_\infty + 1, \ldots, q - d_0\}$.

Notice that the I'_{0+} columns of $\begin{bmatrix} \mathbf{A}_\infty^= \\ \vec{1} \end{bmatrix}$ are linearly independent (because their $\mathbf{A}_\infty^=$ parts are already linearly independent). Let J denote the indices to columns of $\begin{bmatrix} \mathbf{A}_\infty^= \\ \vec{1} \end{bmatrix}$ that are linearly independent with the I'_{0+} columns. That is,

$$J = \left\{i_+ \in ([q] \setminus I'_{0+}) : \text{the } I'_{0+} \cup \{i_+\} \text{ columns of } \begin{bmatrix} \mathbf{A}_\infty^= \\ \vec{1} \end{bmatrix} \text{ are linearly independent}\right\} \tag{11}$$

By definition, we have $J \neq \emptyset$, because according to Assumption 2,

$$\text{Rank}\left(\begin{bmatrix} \mathbf{A}_\infty^= \\ \vec{1} \end{bmatrix}\right) = \text{Rank}\left(\mathbf{A}_\infty^=\right) + 1 = q - d_\infty + 1 > q - d_\infty = |I'_{0+}|$$

Next, we define two specific columns: $i_+ \in J$ and i_- in the following two cases ($i_+ = i_-$ in case 2), prove that the $I'_{0+} \cup I'_{0-} \cup \{i_-\}$ columns of $\begin{bmatrix} \mathbf{A}_* \\ \vec{1} \end{bmatrix}$ are linearly independent (in Claim 4), and then use them to define I_{0+}, I_{0-} and I_1.

- **Case 1: $J \cap I'_{0-} \neq \emptyset$.** Let i_+ denote an arbitrary number in $J \cap I'_{0-}$ and let $i_- \in ([q] \setminus (I'_{0+} \cup I'_{0-}))$ denote an arbitrary number such that the $I'_{0+} \cup I'_{0-} \cup \{i_-\}$ columns of $\begin{bmatrix} \mathbf{A}_* \\ \vec{1} \end{bmatrix}$ are linearly independent. The existence of such i_- is guaranteed by the following two observations. First, according to the definitions of I'_{0+} and I'_{0-}, the $I'_{0+} \cup I'_{0-}$ columns of $\begin{bmatrix} \mathbf{A}_* \\ \vec{1} \end{bmatrix}$ are linearly independent. Second, according to Assumption 1,
$$\text{Rank}\left(\begin{bmatrix} \mathbf{A}_* \\ \vec{1} \end{bmatrix}\right) \geq \text{Rank}\left(\begin{bmatrix} \mathbf{A}_0^= \\ \vec{1} \end{bmatrix}\right) = \text{Rank}\left(\mathbf{A}_0^=\right) + 1 = q - d_0 + 1 > q - d_0 = |I'_{0+} \cup I'_{0-}|$$
- **Case 2: $J \cap I'_{0-} = \emptyset$.** Choose any $i_+ \in J \subseteq I_1$ and let $i_- = i_+$. We now prove that the $I'_{0+} \cup I'_{0-} \cup \{i_-\}$ columns of $\begin{bmatrix} \mathbf{A}_* \\ \vec{1} \end{bmatrix}$ are linearly independent. Suppose for the sake of contradiction this is not true, which means that column i_- of $\begin{bmatrix} \mathbf{A}_* \\ \vec{1} \end{bmatrix}$ can be written as an affine combination of the $I'_{0+} \cup I'_{0-}$ columns of $\begin{bmatrix} \mathbf{A}_* \\ \vec{1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_\infty \\ \mathbf{A}_0 \\ \vec{1} \end{bmatrix}$. This mean that column i_- of $\begin{bmatrix} \mathbf{A}_\infty^= \\ \vec{1} \end{bmatrix}$ can be written as the same affine

combination of the $I'_{0+} \cup I'_{0-}$ columns of $\begin{bmatrix} \mathbf{A}_\infty^= \\ \vec{1} \end{bmatrix}$. Recall that $J \cap I'_{0-} = \emptyset$, which means that in matrix $\begin{bmatrix} \mathbf{A}_\infty^= \\ \vec{1} \end{bmatrix}$, each column in I'_{0-} is an affine combination of the I'_{0+} columns of $\begin{bmatrix} \mathbf{A}_\infty^= \\ \vec{1} \end{bmatrix}$. Therefore, in $\begin{bmatrix} \mathbf{A}_\infty^= \\ \vec{1} \end{bmatrix}$, column i_- is linearly *dependent* with the I'_{0+} columns. This contradicts the definition of i_-, which is the same as $i_+ \in J$.

Notice that in both cases, the following claim holds.

Claim 4. *The $I'_{0+} \cup I'_{0-} \cup \{i_-\}$ columns of $\begin{bmatrix} \mathbf{A}_* \\ \vec{1} \end{bmatrix}$ are linearly independent.*

Define I_{0+}, I_{0-}, and I_1. Given I'_{0+}, I'_{0-}, i_+, and i_- defined above, we are now ready to define I_{0+}, I_{0-}, and I_1 as follows. Let

$$I_{0+} = I'_{0+} \cup \{i_+\}, I_{0-} = I'_{0-} \cup \{i_-\} \setminus \{i_+\}, \text{ and } I_1 = [q] \setminus (I_{0+} \cup I_{0-})$$

By definition, we have $|I_{0+}| = q - d_\infty + 1$, $|I_{0+}| = d_0 - d_\infty$, and $|I_1| = d_0 - 1$. For convenience, we rename the coordinates so that

$$I_{0+} = \{1, \ldots, q - d_\infty + 1\}, I_{0-} = \{q - d_\infty + 2, \ldots, q - d_0 + 1\} \text{ and } I_1 = \{q - d_0 + 2, \ldots, q\}$$

Verify Condition 1. Because $i_+ \in J$, the $I_{0+} = I'_{0+} \cup \{i_+\}$ columns of $\begin{bmatrix} \mathbf{A}_\infty^= \\ \vec{1} \end{bmatrix}$ are linearly independent (due to the definition of J in (11)), which means that Condition 1 is satisfied.

Verify Condition 2. Recall that when defining i_-, we proved that the $I_{0+} \cup I_{0-} = I'_{0+} \cup I'_{0-} \cup \{i\}$ columns of $\begin{bmatrix} \mathbf{A}_* \\ \vec{1} \end{bmatrix}$ are linearly independent. Because $\text{Rank}\left(\begin{bmatrix} \mathbf{A}_\infty^= \\ \mathbf{A}_0^= \end{bmatrix}\right) = \text{Rank}(\mathbf{A}_*) = \text{Rank}(\mathbf{A}_0^=) = q - d_0$, each row in $\mathbf{A}_\infty^=$ can be written as an affine combination of rows in $\mathbf{A}_0^=$. Therefore, if some linear combination of the $I_{0+} \cup I_{0-} = I'_{0+} \cup I'_{0-} \cup \{i\}$ columns of $\begin{bmatrix} \mathbf{A}_0^= \\ \vec{1} \end{bmatrix}$ equals to $\vec{0}$, then the same linear combination of the $I_{0+} \cup I_{0-} = I'_{0+} \cup I'_{0-} \cup \{i\}$ columns of $\begin{bmatrix} \mathbf{A}_* \\ \vec{1} \end{bmatrix}$ equals to $\vec{0}$ as well, which is a contradiction to Claim 4. This verifies Condition 2.

In the remainder of the proof, we let

$$I_0 = I_{0+} \cup I_{0-} \text{ and } I_{1+} = I_1 \cup I_{0-}$$

This leads to two partitions of $[q]$, i.e., $[q] = I_0 \cup I_1 = I_{0+} \cup I_{1+}$, which will be used in the next step.

Step 2 of poly upper bound: Bound the width of coordinates in I_{0+} and I_0. For any polyhedron $\mathcal{H} \subseteq \mathbb{R}^q$, any $I \subseteq [q]$, any $\vec{y}_I \in \mathbb{R}^I$, and any $i \in ([q] \setminus I)$, let $\text{width}_i(\mathcal{H}, \vec{y}_I)$ denote the difference between the maximum value of the i-th component of vectors in $\mathcal{H}$ whose I-components are $\vec{y}_I$ and the minimum value of the i-th component of vectors in $\mathcal{H}$ whose I-components are $\vec{y}_I$. Formally,

$$\text{width}_i(\mathcal{H}, \vec{y}_I) = \max_{\vec{x} \in \mathcal{H}: [\vec{x}]_I = \vec{y}_I} [\vec{x}]_i - \min_{\vec{x} \in \mathcal{H}: [\vec{x}]_I = \vec{y}_I} [\vec{x}]_i$$

Recall that $\mathcal{H}_{B,n} = \{\vec{x} \in \mathcal{H}_B : \vec{x} \cdot \vec{1} = n\}$. In Step 2.1 and Step 2.2, we bound $\text{width}_i(\mathcal{H}_{B,n}, \vec{y}_I)$ for $I = I_{1+} = I_{0-} \cup I_1$ and $I = I_1$, respectively.

Step 2.1 of poly upper bound: Bound the width of coordinates in I_{0+}. In this step, we prove that there exists a constant C^* such that for any B, any n, any $\vec{y}_{I_{1+}} \in \mathbb{R}^{I_{1+}}$, and any $i \in I_{0+}$,

$$\text{width}_i(\mathcal{H}_{B,n}, \vec{y}_{I_{1+}}) \le C^*$$

Notice that $\mathcal{H}_{B,n} \subseteq \mathcal{H}_{\infty,n} = \{\vec{x} \in \mathcal{H}_\infty : \vec{x} \cdot \vec{1} = n\}$. Therefore, it suffices to prove the following stronger inequality.

$$\text{width}_i(\mathcal{H}_{\infty,n}, \vec{y}_{I_{1+}}) \le C^* \tag{12}$$

According to the V-representation of $\mathcal{H}_\infty$, for any $\vec{y} = (\vec{y}_{I_{0+}}, \vec{y}_{I_{1+}}) \in \mathcal{H}_{\infty,n} \subseteq \mathcal{H}_\infty$, we can write $\vec{y} = \vec{v} + \vec{x}$, where $\vec{v} = (\vec{v}_{I_{0+}}, \vec{v}_{I_{1+}})$ is in a convex polytope and $\vec{x} = (\vec{x}_{I_{0+}}, \vec{x}_{I_{1+}})$ is in the characteristic cone of $\mathcal{H}_\infty$. Let $n' = \vec{x} \cdot 1$. Next, we use Gauss-Jordan elimination to define a matrix $\mathbf{D}_\infty$ based on $\begin{bmatrix} \mathbf{A}_\infty^= \\ \vec{1} \end{bmatrix}$, such that

$$\vec{x}_{I_{0+}} = (\vec{x}_{I_{1+}}, n') \times \mathbf{D}_\infty \tag{13}$$

By Claim 3, we have $\vec{x} \in \mathcal{C}_\infty \subseteq \mathcal{P}_\infty$. Recall from Condition 1 that the first $q - d_\infty + 1$ columns (i.e., the I_{0+} columns) of $\begin{bmatrix} \mathbf{A}_\infty^= \\ \vec{1} \end{bmatrix}$ are linearly independent, and recall from Assumption 2 that $\text{Rank}\left(\begin{bmatrix} \mathbf{A}_\infty^= \\ \vec{1} \end{bmatrix}\right) = q - d_\infty + 1$. Therefore, Gauss-Jordan elimination on

$\left[\begin{array}{c}\mathbf{A}_\infty^=\\ \vec{1}\end{array}\right] \times (\vec{x})^\top = \left(\vec{0}, n'\right)^\top$ leads to a $d_\infty \times (q - d_\infty + 1)$ matrix $\mathbf{D}_\infty$ such that

$$\left[\begin{array}{c}\mathbf{A}_\infty^=\\ \vec{1}\end{array}\right] \times (\vec{x})^\top = \left(\vec{0}, n'\right)^\top \text{ if and only if } \vec{x}_{I_0+} = (\vec{x}_{I_{1+}}, n') \times \mathbf{D}_\infty,$$

which proves (13).

Let $C_{\max}$ denote the maximum L_∞ norm of vectors in $\mathcal{V}$, which means that $|\vec{v}|_\infty \le C_{\max}$. Then,

$$\left|n - n'\right| = \left|\vec{y} \cdot \vec{1} - \vec{x} \cdot \vec{1}\right| = |\vec{v} \cdot \vec{1}| \le qC_{\max}$$

Let $\hat{C}$ denote the maximum absolute value of entries in $\mathbf{D}_\infty$ and let $C^* = C_{\max} + 2qC_{\max}\hat{C}$. We prove that $\vec{y}_{I_{0+}}$ is in a C^* neighborhood of $(\vec{y}_{I_{1+}}, n) \times \mathbf{D}_\infty$ in L_∞ as follows.

$$\begin{aligned}
&|\vec{y}_{I_{0+}} - (\vec{y}_{I_{1+}}, n) \times \mathbf{D}_\infty|_\infty = |\vec{v}_{I_{0+}} + \vec{x}_{I_{0+}} - (\vec{v}_{I_{1+}} + \vec{x}_{I_{1+}}, n) \times \mathbf{D}_\infty|_\infty \\
=&|\vec{v}_{I_{0+}} + (\vec{x}_{I_{1+}}, n') \times \mathbf{D}_\infty - (\vec{v}_{I_{1+}} + \vec{x}_{I_{1+}}, n) \times \mathbf{D}_\infty|_\infty && \text{by (13)}\\
=&|\vec{v}_{I_{0+}} - (\vec{v}_{I_{1+}}, n - n') \times \mathbf{D}_\infty|_\infty \\
\le& C_{\max} + 2qC_{\max}\hat{C} = C^*
\end{aligned}$$

This proves (12) and completes Step 2.1.

Step 2.2 of poly upper bound: Bound the width of coordinates in I_0. In this step, we prove that there exists a constant C^* such that for any $B \ge 0$, any n, any $\vec{y}_{I_1} \in \mathbb{R}^{I_1}$, and any $i \in I_0$,

$$\text{width}_i(\mathcal{H}_{B,n}, \vec{y}_{I_1}) \le C^*(B+1) \tag{14}$$

We first prove that for any $\vec{x} \in \mathcal{H}_B$, there exists $\vec{x}' \in \mathcal{C}_0$ that is $O(B+1)$ away from $\vec{x}$ in L_∞.

Claim 5. *There exists C such that for any $\vec{x} \in \mathcal{H}_B$, there exists $\vec{x}' \in \mathcal{C}_0$ such that $|\vec{x} - \vec{x}'|_\infty \le C(B+1)$.*

Proof. The proof is done by analyzing the following two linear programs, denoted by $\text{LP}^B_{\mathcal{H}}$ and $\text{LP}^B_{\text{Cone}}$ whose variables are $\vec{x}$ and $\vec{o}$.

$\text{LP}^B_{\mathcal{H}}$	$\text{LP}^B_{\text{Cone}}$
max 0	max 0
s.t. $\mathbf{A}_S \times (\vec{x})^\top \le (\mathbf{b}_S)^\top$	s.t. $\mathbf{A}_S \times (\vec{x})^\top \le \left(\vec{0}\right)^\top$
$\mathbf{A}_T \times (\vec{x} + \vec{o} \times \mathbf{O})^\top \le (\mathbf{b}_T)^\top$	$\mathbf{A}_T \times (\vec{x} + \vec{o} \times \mathbf{O})^\top \le \left(\vec{0}\right)^\top$
$-\vec{o} \le \vec{0}$	$-\vec{o} \le \vec{0}$
$\vec{c} \cdot \vec{o} \le B$	$\vec{c} \cdot \vec{o} \le 0$

Because $\vec{x} \in \mathcal{H}_B$, there exists $\vec{o} \ge \vec{0}$ such that $\vec{x} + \vec{o} \times \mathbf{O} \in \mathcal{H}_T$ and $\vec{c} \cdot \vec{x} \le B$. Therefore, $(\vec{x}, \vec{w})$ is a feasible solution to $\text{LP}^B_{\mathcal{H}}$. Notice that $\text{LP}^B_{\text{Cone}}$ is feasible (for example $\vec{0}$ is a feasible solution) and $\text{LP}^B_{\mathcal{H}}$ and $\text{LP}^B_{\text{Cone}}$ only differ on the right hand side of the inequalities. Therefore, due to [15, Theorem 5 (i)], there exists a feasible solution $(\vec{x}', \vec{w}')$ to $\text{LP}^B_{\text{Cone}}$ that is no more than $q\Delta \max\{|\mathbf{b}_S - \vec{0}|_\infty, |\mathbf{b}_T - \vec{0}|_\infty, B\} = O(B+1)$ away from $(\vec{x}, \vec{w})$ in L_∞, where Δ is the maximum absolute value of determinants of square sub-matrices of the left hand side of $\text{LP}^B_{\mathcal{H}}$ and $\text{LP}^B_{\text{Cone}}$, i.e., $\left[\begin{array}{cc}\mathbf{A}_S & 0\\ \mathbf{A}_T & \mathbf{A}_T \times (\mathbf{O})^\top\\ 0 & -\mathbb{I}\\ 0 & \vec{c}\end{array}\right]$. Recall that $\vec{c} \ge \vec{0}$. This means that the $-\vec{o} \le \vec{0}$ constraint and the $\vec{c} \cdot \vec{o} \le 0$ constraint in $\text{LP}^B_{\text{Cone}}$ imply that $\vec{o} = \vec{0}$, which means that $\vec{x}' \in \mathcal{C}_0$. This completes the proof of Claim 5. □

Like in Step 2.1, we define $\mathbf{D}_0$ to be the matrix obtained from applying Gauss-Jordan elimination on $\left[\begin{array}{c}\mathbf{A}_0^=\\ \vec{1}\end{array}\right]$. That is, for every $\vec{x}' = (\vec{x}'_{I_0}, \vec{x}'_{I_1}) \in \mathcal{C}_0$, let $n' = \vec{x}' \cdot \vec{1}$, we have

$$\vec{x}'_{I_0} = (\vec{x}'_{I_1}, n') \times \mathbf{D}_0 \tag{15}$$

Next, we use Claim 5 to prove that for any $\vec{y} = (\vec{y}_{I_0}, \vec{y}_{I_1}) \in \mathcal{H}_{B,n}$, $|\vec{y} - ((\vec{y}_{I_1}, n) \times \mathbf{D}_0, \vec{y}_{I_1})|_\infty = O(B+1)$, which would prove (14).

For any $\vec{y} = (\vec{y}_{I_0}, \vec{y}_{I_1}) \in \mathcal{H}_{B,n}$, let $\vec{x}' = (\vec{x}'_{I_0}, \vec{x}'_{I_1}) \in \mathcal{C}_0$ denote the vector in $\mathcal{C}_0$ that is no more than $C(B+1)$ away from $\vec{y}$ guaranteed by Claim 5. This means that $\vec{x}'_{I_1}$ and n' are $O(B+1)$ away from $\vec{y}_{I_1}$ and n, respectively, which implies that $\vec{x}'_{I_0} = (\vec{x}'_{I_1}, n') \times \mathbf{D}_0$ is $O(B+1)$ away from $(\vec{y}_{I_1}, n) \times \mathbf{D}_0$. Also because $\vec{x}'_{I_0}$ is $O(B+1)$ away from $\vec{y}_{I_0}$, we have that $\vec{y}_{I_0}$ is $O(B+1)$ away from $(\vec{y}_{I_1}, n) \times \mathbf{D}_0$. Formally, let $d^0_{\max}$ denote the maximum absolute value of entries in $\mathbf{D}_0$, we have the following bound.

$$
\begin{aligned}
&|\vec{y} - ((\vec{y}_{I_1}, n) \times \mathbf{D}_0, \vec{y}_{I_1})|_\infty = |\vec{y}_{I_0} - (\vec{y}_{I_1}, n) \times \mathbf{D}_0|_\infty \\
\leq & |\vec{y}_{I_0} - \vec{x}'_{I_0}|_\infty + |\vec{x}'_{I_0} - (\vec{x}'_{I_1}, n') \times \mathbf{D}_0|_\infty + |(\vec{x}'_{I_1}, n') \times \mathbf{D}_0 - (\vec{y}_{I_1}, n) \times \mathbf{D}_0|_\infty \\
\leq & C(B+1) + 0 + |(\vec{x}'_{I_1} - \vec{y}_{I_1}, n' - n) \times \mathbf{D}_0|_\infty && \text{by (15)} \\
\leq & C(B+1)(2qd^0_{\max} + 1)
\end{aligned}
$$

The last inequality holds because $|\vec{x}'_{I_1} - \vec{y}_{I_1}|_\infty \leq C(B+1)$ and $|n' - n| \leq qC(B+1)$. This completes the proof of Step 2.2.

Step 3 of poly upper bound: Upper-bound the probability. Recall that $\vec{X}_{\vec{\pi}} = \text{Hist}(P)$, where P consists of n independent random variables $Y_1, \ldots, Y_n$ distributed as $\vec{\pi}$. Like [77], we represent each Y_j as two random variables Z_j and W_j and a simple Bayesian network based on the partition $[q] = I_0 \cup I_1$.

Definition 16 (**Alternative representation of $Y_1, \ldots, Y_n$ [76]**). *For each $j \leq n$, we define a Bayesian network with two random variables $Z_j \in \{0, 1\}$ and $W_j \in [q]$, where Z_j is the parent of W_j. The conditional probabilities are defined as follows.*

- *For each $\ell \in \{0, 1\}$, let $\Pr(Z_j = \ell) \triangleq \Pr(Y_j \in I_\ell)$.*
- *For each $\ell \in \{0, 1\}$ and each $t \leq q$, let $\Pr(W_j = t | Z_j = \ell) \triangleq \Pr(Y_j = t | Y_j \in I_\ell)$.*

In particular, if $t \notin I_\ell$ then $\Pr(W_j = t | Z_j = \ell) = 0$. It is not hard to verify that for any $j \leq n$, W_j has the same distribution as Y_j. For any $\vec{z} \in \{0, 1\}^n$, we let $\text{Id}_0(\vec{z}) \subseteq [n]$ denote the indices of components of $\vec{z}$ that equal to 0. Given $\vec{z}$, we define the following random variables.

- Let $\vec{W}_{\text{Id}_0(\vec{z})} \triangleq \{W_j : j \in \text{Id}_0(\vec{z})\}$. That is, $\vec{W}_{\text{Id}_0(\vec{z})}$ consists of random variables $\{W_j : z_j = 0\}$.
- Let $\text{Hist}(\vec{W}_{\text{Id}_0(\vec{z})})$ denote the vector of the $q - d_0 + 1 = |I_0|$ random variables that correspond to the histogram of $\vec{W}_{\text{Id}_0(\vec{z})}$ restricted to I_0. Technically, the domain of every random variable in $\vec{W}_{\text{Id}_0(\vec{z})}$ is $[q]$, but since they only receive positive probabilities on I_0, they are treated as random variables over I_0 when $\text{Hist}(\vec{W}_{\text{Id}_0(\vec{z})})$ is defined.
- Similarly, let $\vec{W}_{\text{Id}_1(\vec{z})} \triangleq \{W_j : j \in \text{Id}_1(\vec{z})\}$ and let $\text{Hist}(\vec{W}_{\text{Id}_1(\vec{z})})$ denote the vector of $|I_1| = d_0 - 1$ random variables that correspond to the histogram of $\vec{W}_{\text{Id}_1(\vec{z})}$.

Let $\mathcal{H}^{\mathbb{Z}}_{B,n} \triangleq \mathcal{H}_{B,n} \cap \mathbb{Z}^q$. For any $\vec{y}_1 \in \mathbb{Z}^{d_0-1}_{\geq 0}$, we let $\mathcal{H}^{\mathbb{Z}}_{B,n}|_{\vec{y}_1}$ denote the I_0 components of $\vec{y} \in \mathcal{H}^{\mathbb{Z}}_{B,n}$ whose I_1 components are $\vec{y}_1$. Formally,

$$\mathcal{H}^{\mathbb{Z}}_{B,n}|_{\vec{y}_1} \triangleq \left\{\vec{y}_0 \in \mathbb{Z}^{q-d_0+1}_{\geq 0} : (\vec{y}_0, \vec{y}_1) \in \mathcal{H}^{\mathbb{Z}}_{B,n}\right\}$$

We recall the following calculations in [77] for any $\vec{\pi} \in \Pi^n$, which is done by first separating the $|\text{Id}_0(\vec{z})| \geq 0.9\epsilon n$ case (which happens with $1 - \exp(-\Omega(n))$ probability) from the $|\text{Id}_0(\vec{z})| < 0.9\epsilon n$ (which happens with exponentially small probability), then applying the law of total probability conditioned on $\vec{Z}$, and finally using the conditional independence in the Bayesian network (i.e., $\vec{W}$'s are independent given $\vec{Z}$) to simplify the formula.

$$
\begin{aligned}
\Pr_{P\sim\vec{\pi}}(\text{Hist}(P) \in \mathcal{H}^{\mathbb{Z}}_{B,n}) \leq & \sum_{\vec{z} \in \{0,1\}^n : |\text{Id}_0(\vec{z})| \geq 0.9\epsilon n} \Pr(\vec{Z} = \vec{z}) \sum_{\vec{y}_1 \in \mathbb{Z}^{d_0-1}_{\geq 0}} \Pr\left(\text{Hist}(\vec{W}_{\text{Id}_1(\vec{z})}) = \vec{y}_1 \;\middle|\; \vec{Z} = \vec{z}\right) \\
& \times \Pr\left(\text{Hist}(\vec{W}_{\text{Id}_0(\vec{z})}) \in \mathcal{H}^{\mathbb{Z}}_{B,n}|_{\vec{y}_1} \;\middle|\; \vec{Z} = \vec{z}\right) + \Pr(|\text{Id}_0(\vec{z})| < 0.9\epsilon n)
\end{aligned}
\tag{16}
$$

To upper-bound (16), we will show that for any $\vec{z}$ with $|\text{Id}_0(\vec{z})| \geq 0.9\epsilon n$ and any $\vec{y}_1 \in \mathbb{Z}^{d_0-1}_{\geq 0}$,

$$\Pr\left(\text{Hist}(\vec{W}_{\text{Id}_0(\vec{z})}) \in \mathcal{H}^{\mathbb{Z}}_{B,n}|_{\vec{y}_1} \;\middle|\; \vec{Z} = \vec{z}\right) = O((B+1)^{d_\Delta}) \times O\left(n^{\frac{d_0-q}{2}}\right) \tag{17}$$

Conditioned on $\vec{Z} = \vec{z}$, $\text{Hist}(\vec{W}_{\text{Id}_0(\vec{z})})$ can be viewed as a PMV of $|\text{Id}_0(\vec{z})|$ strictly positive independent variables over $I_0 = [q - d_0 + 1]$. Therefore, according to the point-wise anti-concentration bound [76, Lemma 3 in the Appendix], for any $\vec{z}$ with $|\text{Id}_0(\vec{z})| \geq 0.9\epsilon n$ and any $\vec{y}_0 \in \mathcal{H}^{\mathbb{Z}}_{B,n}|_{\vec{y}_1}$,

$$\Pr\left(\text{Hist}(\vec{W}_{\text{Id}_0(\vec{z})}) = \vec{y}_0 \;\middle|\; \vec{Z} = \vec{z}\right) = O\left(|\text{Id}_0(\vec{z})|^{\frac{d_0-q}{2}}\right) = O\left(n^{\frac{d_0-q}{2}}\right)$$

Then, to prove (17), it suffices to prove $|\mathcal{H}^{\mathbb{Z}}_{B,n}|_{\vec{y}_1}| = O((B+1)^{d_\Delta})$. This is done by enumerating vectors in $\mathcal{H}^{\mathbb{Z}}_{B,n}|_{\vec{y}_1}$ as follows: According to Step 2.2 of the poly upper bound, each I_{0-} component of vectors in $\mathcal{H}^{\mathbb{Z}}_{B,n}|_{\vec{y}_1}$ has no more than $\lceil C^*(B+1) + 1 \rceil$ choices, and given the I_{0-} components, each I_{0+} component has no more than $\lceil C^* + 1 \rceil$ choices, where C^* is the maximum value of the constants in Step 2.1 and 3.2. Therefore,

$$|\mathcal{H}^{\mathbb{Z}}_{B,n}|_{\vec{y}_1}| \leq (C^* + 1)^{q-d_\infty+1}(C^*(B+1) + 1)^{d_\infty - d_0} = O((B+1)^{d_\infty - d_0}) = O((B+1)^{d_\Delta})$$

This proves (17). Then, combining (16) and (17), and following a similar argument as in [77], we have

$$\begin{aligned}
&\Pr_{P\sim\vec{\pi}}(\mathrm{Hist}(P)\in\mathcal{U}_{n,B}) \leq \Pr_{P\sim\vec{\pi}}(\mathrm{Hist}(P)\in\mathcal{H}^{\mathbb{Z}}_{B,n})\\
\leq &\sum_{\vec{z}\in\{0,1\}^n:|\mathrm{Id}_0(\vec{z})|\geq 0.9\epsilon n}\Pr(\vec{Z}=\vec{z})\sum_{\vec{y}_1\in\mathbb{Z}^{d_0-1}_{\geq 0}}\Pr\left(\mathrm{Hist}(\vec{W}_{\mathrm{Id}_1(\vec{z})})=\vec{y}_1\;\middle|\;\vec{Z}=\vec{z}\right)\\
&\qquad\times O((B+1)^{d_\Delta})\times O\left(n^{\frac{d_0-q}{2}}\right)+\exp(-\Theta(n))\\
=&\Pr(|\mathrm{Id}_0(\vec{Z})|\geq 0.9\epsilon n)\times O((B+1)^{d_\Delta})\times O\left(n^{\frac{d_0-q}{2}}\right)+\exp(-\Theta(n))\\
=&O\left((B+1)^{d_\Delta}\cdot\left(\frac{1}{\sqrt{n}}\right)^{q-d_0}\right)
\end{aligned}$$

This proves the polynomial upper bound for $B\leq\sqrt{n}$.

Proof for the polynomial upper bound of sup for $B>\sqrt{n}$. Notice that for any B and n, $\mathcal{H}_{B,n}\subseteq\mathcal{H}_\infty$. Therefore, it suffices to prove the following stronger claim, which holds for any B, any n, and any $\vec{\pi}\in\Pi^n$.

Claim 6. *For any n and $\vec{\pi}\in\Pi^n$, $\Pr\left(\vec{X}_{\vec{\pi}}\in\mathcal{H}_\infty\right)=O\left((\frac{1}{\sqrt{n}})^{q-d_\infty}\right)$.*

Proof. The proof is similar to Step 3 of the $B\leq\sqrt{n}$ case above. We will define a Bayesian network for the partition $[q]=I_{0+}\cup I_{1+}$, and then apply (12) to enumerate vectors in $\mathcal{H}^{\mathbb{Z}}_{\infty,n}|_{\vec{y}_{1+}}$.

Definition 17 (**Alternative representation of $Y_1,\ldots,Y_n$**). *For each $j\leq n$, we define a Bayesian network with two random variables $Z^+_j\in\{0,1\}$ and $W^+_j\in[q]$, where Z^+_j is the parent of W^+_j. The conditional probabilities are defined as follows.*

- *For each $\ell\in\{0,1\}$, let $\Pr(Z^+_j=\ell)\triangleq\Pr(Y_j\in I_{\ell+})$.*
- *For each $\ell\in\{0,1\}$ and each $t\leq q$, let $\Pr(W^+_j=t|Z^+_j=\ell)\triangleq\Pr(Y_j=t|Y_j\in I_{\ell+})$.*

In particular, if $t\notin I_{\ell+}$ then $\Pr(W^+_j=t|Z^+_j=\ell)=0$. It is not hard to verify that for any $j\leq n$, W^+_j follows the same distribution as Y_j. For any $\vec{z}\in\{0,1\}^n$, we let $\mathrm{Id}_{0+}(\vec{z})\subseteq[n]$ denote the indices of components of $\vec{z}$ that equal to 0. Given $\vec{z}$, we define the following random variables.

- Let $\vec{W}^+_{\mathrm{Id}_{0+}(\vec{z})}\triangleq\{W^+_j:j\in\mathrm{Id}_{0+}(\vec{z})\}$. That is, $\vec{W}^+_{\mathrm{Id}_{0+}(\vec{z})}$ consists of random variables $\{W^+_j:z_j=0\}$.
- Let $\mathrm{Hist}(\vec{W}^+_{\mathrm{Id}_{0+}(\vec{z})})$ denote the vector of the $|I_{0+}|=q-d_\infty+1$ random variables that correspond to the histogram of $\vec{W}^+_{\mathrm{Id}_{0+}(\vec{z})}$ restricted to I_{0+}.
- Similarly, let $\vec{W}^+_{\mathrm{Id}_{1+}(\vec{z})}\triangleq\{W^+_j:j\in\mathrm{Id}_{1+}(\vec{z})\}$ and let $\mathrm{Hist}(\vec{W}^+_{\mathrm{Id}_{1+}(\vec{z})})$ denote the vector of $|I_{1+}|=d_\infty-1$ random variables that correspond to the histogram of $\vec{W}^+_{\mathrm{Id}_{1+}(\vec{z})}$.

For any $\vec{y}_{1+}\in\mathbb{Z}^{I_{1+}}_{\geq 0}$, we let $\mathcal{H}^{\mathbb{Z}}_{\infty,n}|_{\vec{y}_{1+}}$ denote the I_{0+} components of $\vec{y}\in\mathcal{H}^{\mathbb{Z}}_{\infty,n}$ whose I_{1+} components are $\vec{y}_{1+}$. Formally,

$$\mathcal{H}^{\mathbb{Z}}_{\infty,n}|_{\vec{y}_{1+}}\triangleq\left\{\vec{y}_{0+}\in\mathbb{Z}^{q-d_0+1}_{\geq 0}:(\vec{y}_{0+},\vec{y}_{1+})\in\mathcal{H}^{\mathbb{Z}}_{\infty,n}\right\}$$

Like Step 3 of the $B\leq\sqrt{n}$ case, for any $\vec{\pi}\in\Pi^n$,

$$\begin{aligned}
&\Pr_{P\sim\vec{\pi}}(\mathrm{Hist}(P)\in\mathcal{H}^{\mathbb{Z}}_{\infty,n})\\
\leq&\sum_{\vec{z}\in\{0,1\}^n:|\mathrm{Id}_{0+}(\vec{z})|\geq 0.9\epsilon n}\Pr(\vec{Z}^+=\vec{z})\sum_{\vec{y}_{1+}\in\mathbb{Z}^{d_\infty-1}_{\geq 0}}\Pr\left(\mathrm{Hist}(\vec{W}^+_{\mathrm{Id}_{1+}(\vec{z})})=\vec{y}_{1+}\;\middle|\;\vec{Z}^+=\vec{z}\right)\\
&\times\Pr\left(\mathrm{Hist}(\vec{W}^+_{\mathrm{Id}_{0+}(\vec{z})})\in\mathcal{H}^{\mathbb{Z}}_{\infty,n}|_{\vec{y}_{1+}}\;\middle|\;\vec{Z}^+=\vec{z}\right)+\Pr(|\mathrm{Id}_{0+}(\vec{z})|<0.9\epsilon n)
\end{aligned}\tag{18}$$

Following (12) and the point-wise anti-concentration bound [76, Lemma 3 in the Appendix], for any $\vec{z}$ with $|\mathrm{Id}_{0+}(\vec{z})|\geq 0.9\epsilon n$ and any $\vec{y}_{1+}\in\mathbb{Z}^{d_\infty-1}_{\geq 0}$, we have

$$\Pr\left(\mathrm{Hist}(\vec{W}^+_{\mathrm{Id}_{0+}(\vec{z})})\in\mathcal{H}^{\mathbb{Z}}_{\infty,n}|_{\vec{y}_{1+}}\;\middle|\;\vec{Z}^+=\vec{z}\right)=O(1)\times O\left(n^{\frac{d_\infty-q}{2}}\right)\tag{19}$$

Then, combining (18) and (19) and recalling that $\text{Hist}(P)$ is a size-n non-negative integer vector, we have

$$\begin{aligned}
&\Pr_{P\sim\vec{\pi}}(\text{Hist}(P)\in\mathcal{U}_{n,B})\leq\Pr_{P\sim\vec{\pi}}(\text{Hist}(P)\in\mathcal{H}^{\mathbb{Z}}_{\infty,n})\\
\leq&\sum_{\vec{z}\in\{0,1\}^n:|\text{Id}_{0+}(\vec{z})|\geq 0.9\epsilon n}\Pr(\vec{Z}^+=\vec{z})\sum_{\vec{y}_{1+}\in\mathbb{Z}^{d_\infty-1}_{\geq 0}}\Pr\left(\text{Hist}(\vec{W}^+_{\text{Id}_{1+}(\vec{z})})=\vec{y}_{1+}\,\middle|\,\vec{Z}^+=\vec{z}\right)\\
&\times O\left(n^{\frac{d_\infty-q}{2}}\right)+\exp(-\Theta(n))\\
=&\Pr(|\text{Id}_{0+}(\vec{Z}^+)|\geq 0.9\epsilon n)\times O\left(n^{\frac{d_\infty-q}{2}}\right)+\exp(-\Theta(n))=O\left(\left(\frac{1}{\sqrt{n}}\right)^{q-d_\infty}\right)
\end{aligned}$$

This proves Claim 6. □

The polynomial upper bound for $B>\sqrt{n}$ follows after Claim 6.

Proof for the $\Omega\left(\frac{\min\{B+1,\sqrt{n}\}^{d_\Delta}}{(\sqrt{n})^{q-d_0}}\right)$ lower bound of PT-$\Theta(\sqrt{n})$-sup. It suffices to prove the lower bound for $B\leq\sqrt{n}$, because when $B\geq\sqrt{n}$, we have

$$\sup_{\vec{\pi}\in\Pi^n}\Pr(\vec{X}_{\vec{\pi}}\in\mathcal{U}_{n,B})\geq\sup_{\vec{\pi}\in\Pi^n}\Pr(\vec{X}_{\vec{\pi}}\in\mathcal{U}_{n,\sqrt{n}})=\Omega\left((\frac{1}{\sqrt{n}})^{q-d_\infty}\right),$$

which is the desired lower bound.

The proof for $B\leq\sqrt{n}$ proceeds in three steps. In Step 1, for any strictly positive $\pi^*\in\mathcal{C}_0$, we identify an $O(\sqrt{n})$ neighborhood of $n\cdot\pi^*$ that contains $\Omega\left((B+1)^{d_\Delta}\cdot(\sqrt{n})^{d_0-1}\right)$ integer vectors in $\mathcal{U}_{n,B}$. In Step 2, for any $\pi^*\in\text{CH}(\Pi)$, we define $\vec{\pi}^\circ=(\pi^\circ_1,\ldots,\pi^\circ_n)\in\Pi^n$ such that $\sum_{j=1}^n\pi^\circ_j$ is $O(\sqrt{n})$ away from $n\cdot\pi^*$. The lower bound is then proved in Step 3. Among the three steps, Step 1 is the most involved part, and Steps 2 and 3 follow after similar proofs in [77].

Step 1 of poly lower bound: Identify $\Omega\left((B+1)^{d_\Delta}\cdot(\sqrt{n})^{d_0-1}\right)$ vectors in $\mathcal{U}_{n,B}$.

Overview of Step 1. The proof proceeds in four sub-steps. In Step 1.1, we prove in Claim 7 that there exists a constant C' such that for every (possibly non-integer) non-negative vector in $\mathcal{H}_{B,n}$, there is an "accompany" integer vector in $\mathcal{U}_{n,B}$ that is at most C' away in L_∞. Then, we identify a set $\mathcal{R}_{B,n}$ of possibly non-integer vectors in $\mathcal{H}_{B,n}$ that are at least $2C'$ away from each other, and apply Claim 7 to obtain a set $\mathcal{R}^{\mathbb{Z}}_{B,n}$ of integer vectors in $\mathcal{U}_{n,B}$ of the same size (as $\mathcal{R}_{B,n}$). To define $\mathcal{R}_{B,n}$, we explore two directions in an $O(\sqrt{n})$ neighborhood of $n\cdot\pi^*$: the $\mathcal{P}_0$ direction and the $\mathcal{P}_\infty$ direction (which we recall from (10) are affine hulls of $\mathcal{C}_0$ and $\mathcal{C}_\infty$, respectively). More precisely, we will first enumerate $\Omega((\sqrt{n})^{d_0-1})$ vectors in a $\mathcal{P}_0$ neighborhood of $n\cdot\pi^*$ (defined as $\mathcal{R}^0_n$ in Step 1.2), and then enumerate $\Omega((B+1)^{d_\Delta})$ vectors in a neighborhood of $n\cdot\pi^*$ (defined as $\mathcal{R}^\infty_B$ in Step 1.3) that is a complement of $\mathcal{P}_0$ in $\mathcal{P}_\infty$. Finally, in Step 1.4, we formally define $\mathcal{R}_{B,n}$ and $\mathcal{R}^{\mathbb{Z}}_{B,n}$ and prove that $|\mathcal{R}_{B,n}|=|\mathcal{R}^{\mathbb{Z}}_{B,n}|=\Omega((B+1)^{d_\Delta}\cdot(\sqrt{n})^{d_0-1})$ and $\mathcal{R}_{B,n}\subseteq\mathcal{H}_{B,n}\cap\mathbb{R}^q_{\geq 0}$ in Claim 8 and Claim 9, respectively.

Step 1.1: Define C'. We prove the following claim.

Claim 7. *There exists a constant C' that does not depend on n or B, such that for any $\vec{x}\in\mathcal{H}_{B,n}\cap\mathbb{R}^q_{\geq 0}$, there exists a (non-negative integer) vector $\vec{x}'\in\mathcal{U}_{n,B}$ such that $|\vec{x}-\vec{x}'|_\infty<C'$.*

Proof. Consider the following linear program $\text{LP}_{B,n}$ whose variables are $\vec{x}$ and $\vec{o}$:

$$LP_{B,n}=\begin{cases}\max & 0\\ \text{s.t.} & \mathbf{A}_{\text{S}}\times(\vec{x})^\top\leq(\mathbf{b}_{\text{S}})^\top\\ & \mathbf{A}_{\text{T}}\times(\vec{x}+\vec{o}\times\mathbf{O})^\top\leq(\mathbf{b}_{\text{T}})^\top\\ & \vec{x}\cdot\vec{1}=n\\ & \vec{c}\cdot\vec{o}\leq B\\ & \vec{x}\geq\vec{0},\vec{o}\geq\vec{0}\end{cases}$$

It is not hard to verify that, because $\vec{x}\in\mathcal{H}_{B,n}\cap\mathbb{R}^q_{\geq 0}$, $\text{LP}_{B,n}$ has a feasible solution $(\vec{x},\vec{o})$ for some $\vec{o}\geq\vec{0}$ (which may not be an integer vector). Recall that $\mathcal{U}_{n,B}\neq\emptyset$, which means that $\text{LP}_{B,n}$ has a feasible integer solution. Therefore, by [15, Theorem 1], $\text{LP}_{B,n}$ has an integer solution $(\vec{x}',\vec{o}')$ whose L_∞ distance to $(\vec{x},\vec{o})$ is no more than $(q+|\mathbb{O}|)\Delta$, where Δ is the maximum absolute determinant of square submatrices of the matrix that defines $\mathrm{LP}_{B,n}$, i.e., $\begin{bmatrix} \mathbf{A}_{\mathrm{S}} & \vec{0} \\ \mathbf{A}_{\mathrm{T}} & \mathbf{A}_{\mathrm{T}} \times (\mathbf{O})^\top \\ \vec{1} & \vec{0} \\ -\vec{1} & \vec{0} \\ \vec{0} & \vec{c} \\ -\mathbb{I} & \vec{0} \\ \vec{0} & -\mathbb{I} \end{bmatrix}$. Notice that this matrix does not depend on B or n. The claim follows after letting $C' \triangleq (q + |\mathbb{O}|)\Delta$ and noticing that $(\vec{x}', \vec{o}\,') \in \mathcal{U}_{n,B}$. □

Step 1.2: Define $\mathcal{R}_n^0$. Let $\mathcal{P}_0^*$ denote the size-0 vectors of $\mathcal{P}_0$. That is,

$$\mathcal{P}_0^* = \mathcal{P}_0 \cap \left\{\vec{x} : \vec{x} \cdot \vec{1} = 0\right\} = \left\{\vec{x} \in \mathbb{R}^q : \begin{bmatrix} \mathbf{A}_0^{=} \\ \vec{1} \end{bmatrix} \times (\vec{x})^\top = \left(\vec{0}\right)^\top\right\}$$

Recall from Assumption 1 that $\mathrm{Rank}\left(\begin{bmatrix} \mathbf{A}_0^{=} \\ \vec{1} \end{bmatrix}\right) = \mathrm{Rank}(\mathbf{A}_0^{=}) + 1 = q - d_0 + 1$, which means that $\dim(\mathcal{P}_0^*) = q - \mathrm{Rank}\left(\begin{bmatrix} \mathbf{A}_0^{=} \\ \vec{1} \end{bmatrix}\right) = d_0 - 1$. Therefore, $\mathcal{P}_0^*$ contains a basis of $d_0 - 1$ linearly independent vectors, denoted by $\mathbf{B}^0 = \{\vec{p}_1^{\,0}, \ldots, \vec{p}_{d_0-1}^{\,0}\}$. Let $\mathbb{L}^0$ denote the lattice generated by $\mathbf{B}^0$ excluding $\vec{0}$. That is,

$$\mathbb{L}^0 = \left\{\sum\nolimits_{i=1}^{d_0-1} \gamma_i \cdot \vec{p}_i^{\,0} : \forall i \le d_0 - 1, \gamma_i \in \mathbb{Z} \text{ and } \exists i \le d_0 - 1 \text{ s.t. } \gamma_i \neq 0\right\}$$

Let C^0 denote the minimum L_∞ norm of all vectors in $\mathbb{L}^0$, i.e.,

$$C^0 = \inf\left\{|\vec{x}|_\infty : \vec{x} \in \mathbb{L}^0\right\}$$

Next, we prove $C^0 > 0$. Suppose for the sake of contradiction that $C^0 = 0$. Then, there exist a sequence of vectors

$$\left\{\vec{x}^j = \sum\nolimits_{i=1}^{d_0-1} \gamma_i^j \cdot \vec{p}_i^{\,0} : j \in \mathbb{N}\right\}$$

such that for all $j \in \mathbb{N}$, $|\vec{x}^j|_\infty \le \frac{1}{j}$. Let $\vec{\gamma}^j = (\gamma_1^j, \ldots, \gamma_{d_0-1}^j)$. Because $|\vec{\gamma}^j|_\infty \ge 1$, we have $\frac{\vec{x}^j}{|\vec{\gamma}^j|_\infty} \le \frac{1}{j}$. Notice that $\{\vec{\gamma} \in \mathbb{R}^{d_0-1} : |\vec{\gamma}|_\infty = 1\}$ is closed and compact, there exists a subsequence of $\left\{\frac{\vec{\gamma}^j}{|\vec{\gamma}^j|_\infty}\right\}$ that converges to a vector $\vec{\gamma}^*$ with $|\vec{\gamma}^*|_\infty = 1$. Let $\vec{x}^* = \sum_{i=1}^{d_0-1} \gamma_i^* \cdot \vec{p}_i^{\,0}$. It follows that $\left\{\frac{\vec{x}^j}{|\vec{\gamma}^j|_\infty} : j \in \mathbb{N}\right\}$ converges to $\vec{x}^*$, which means that $|\vec{x}^*|_\infty = \lim_{j\to\infty}\left|\frac{\vec{x}^j}{|\vec{\gamma}^j|_\infty}\right|_\infty = 0$. This contradicts the linear independence of vectors in $\mathbf{B}^0$.

Then, we use C^0 and $\mathbf{B}^0$ to define a subset of $\frac{2C'}{C^0} \cdot \mathbb{L}^0 \cup \{\vec{0}\}$ as follows.

$$\mathcal{R}_n^0 = \left\{\sum\nolimits_{i=1}^{d_0-1} \frac{2C'}{C^0}\gamma_i \cdot \vec{p}_i^{\,0} : \forall i \le d_0 - 1, \gamma_i \in \{0, 1, \ldots, \lfloor\sqrt{n}\rfloor\}\right\} \tag{20}$$

By definition, $\mathcal{R}_n^0 \subseteq \mathcal{P}_0^* \subseteq \mathcal{P}_0$, and for every vector $\vec{x} \in \mathcal{R}_n^0$, we have $\vec{x} \cdot \vec{1} = 0$ and $|\vec{x}|_\infty = O(\sqrt{n})$. The value $\frac{2C'}{C^0}$ is chosen so that the L_∞ distance between any pair of different vectors in $\mathcal{R}_n^0$ is at least $2C'$.

Step 1.3: Define $\mathcal{R}_B^\infty$. Recall that $\mathcal{P}_0 \subseteq \mathcal{P}_\infty$, $d_0 = \dim(\mathcal{P}_0)$ and $d_\infty = \dim(\mathcal{C}_\infty) = \dim(\mathcal{P}_\infty)$. In the following procedure, we define a set of d_Δ linear independent vectors $\mathbf{B}^\infty \subseteq \mathcal{C}_\infty$, which are the basis of a complement of $\mathcal{P}_0$ in $\mathcal{P}_\infty$.

Procedure for defining $\mathbf{B}^\infty$. Let $\{\vec{p}_1, \ldots, \vec{p}_{d_\infty}\}$ denote an arbitrary set of d_∞ linearly independent vectors in $\mathcal{C}_\infty$, whose existence is guaranteed by the fact that $\dim(\mathcal{C}_\infty) = d_\infty$. Start with $\mathbf{B}^\infty = \emptyset$. For every $1 \le j \le d_\infty$, we add $\vec{p}_j$ to $\mathbf{B}^\infty$ if and only if it is linearly independent with $\mathcal{P}_0 \cup \mathbf{B}^\infty$. At the end of procedure we have $|\mathbf{B}^\infty| = d_\infty - d_0 = d_\Delta$. W.l.o.g., let $\mathbf{B}^\infty = \{\vec{p}_1^{\,\infty}, \ldots, \vec{p}_{d_\Delta}^{\,\infty}\}$.

Let $\overline{\mathcal{P}_0} = \mathrm{Span}(\mathbf{B}^\infty)$. It follows that $\dim(\overline{\mathcal{P}_0}) = d_\Delta$ and

$$\mathcal{P}_0 \cap \overline{\mathcal{P}_0} = \{\vec{0}\} \text{ and } \mathcal{P}_0 + \overline{\mathcal{P}_0} = \mathcal{P}_\infty \tag{21}$$

For each $i \le d_\Delta$, because $\vec{p}_i^{\,\infty} \in \mathcal{C}_\infty$, we have $\vec{p}_i^{\,\infty} \in \mathcal{H}_{\mathrm{S},\le 0}$ and we can write $\vec{p}_i^{\,\infty} = \vec{y}_i^{\,\infty} - \vec{o}_i^{\,\infty} \times \mathbf{O}$, where $\vec{y}_i^{\,\infty} \in \mathcal{H}_{\mathrm{T},\le 0}$ and $\vec{o}_i^{\,\infty} \ge \vec{0}$. For every $i \le d_\Delta$, we must have $\vec{w}_i^{\,\infty} \neq \vec{0}$, because otherwise $\vec{p}_i^{\,\infty} \in \mathcal{C}_0 \subseteq \mathcal{P}_0$, which contradicts (21), because $\vec{p}_i^{\,\infty} \in \overline{\mathcal{P}_0}$. Recall that $\vec{c} > \vec{0}$. W.l.o.g. we can assume $\vec{c} \cdot \vec{w}_i^{\,\infty} = 1$, otherwise we divide $\vec{p}_i^{\,\infty}$ by $\vec{c} \cdot \vec{w}_i^{\,\infty}$.

Let $\mathbb{L}^\infty$ denote the lattice generated by $\mathbf{B}^\infty$ excluding $\vec{0}$. That is,

$$\mathbb{L}^\infty = \left\{\sum\nolimits_{i=1}^{d_\Delta} \eta_i \cdot \vec{p}_i^{\,\infty} : \forall i \le d_\Delta, \eta_i \in \mathbb{Z} \text{ and } \exists i \le d_\Delta \text{ s.t. } \eta_i \neq 0\right\}$$

Let C^∞ denote the minimum distance between $\mathbb{L}^\infty$ and $\mathcal{P}_0$. That is,

$$C^\infty = \inf\left\{\left|\vec{x}^{\,\infty} - \vec{y}\right|_\infty : \vec{x}^{\,\infty} \in \mathbb{L}^\infty, \vec{y} \in \mathcal{P}_0\right\} \tag{22}$$

Following an argument that is similar to the proof of $C^0 > 0$, we have $C^\infty > 0$. Formally, for the sake of contradiction suppose $C^\infty = 0$. Then, there exist two sequences of vectors

$$\left\{\vec{x}^j = \sum\nolimits_{i=1}^{d_\Delta} \eta_i^j \cdot \vec{p}_i^{\,\infty} : j \in \mathbb{N}\right\} \text{ and } \left\{\vec{y}^j \in \mathcal{P}_0 : j \in \mathbb{N}\right\},$$

such that for all $j \in \mathbb{N}$, $|\vec{x}^j - \vec{y}^j|_\infty \leq \frac{1}{j}$. Let $\vec{\eta}^j = (\eta_1^j, \ldots, \eta_{d_\Delta}^j)$. Because $|\vec{\eta}^j|_1 \geq 1$ and $\mathcal{P}_0$ includes $\vec{0}$, the distance between $\frac{\vec{x}^j}{|\vec{\eta}^j|_1}$ and $\mathcal{P}_0$ is at most $\frac{1}{j}$. Notice that $\{\vec{\eta} \in \mathbb{R}^{d_\Delta} : |\vec{\eta}|_1 = 1\}$ is closed and compact, there exists a subsequence of $\left\{\frac{\vec{\eta}^j}{|\vec{\eta}^j|_1}\right\}$ that converges to a vector $\vec{\eta}^*$ with $|\vec{\eta}^*|_1 = 1$. Let $\{j_i : i \in \mathbb{N}\}$ denote the indices of the subsequence. Let $\vec{x}^* = \sum_{i=1}^{d_\Delta} \eta_i^* \cdot \vec{p}_i^{\,\infty}$. Because $\vec{p}_i^{\,\infty}$'s are linearly independent, we have $\vec{x}^* \neq \vec{0}$. Then, $\{\vec{x}^{j_i} : i \in \mathbb{N}\}$ and $\{\vec{y}^{j_i} : i \in \mathbb{N}\}$ both converge to $\vec{x}^*$. Because $\mathcal{P}_0$ is closed, we have $\vec{x}^* \in \mathcal{P}_0$, which means that $\vec{x}^* \in \overline{\mathcal{P}_0} \cap \mathcal{P}_0 = \{\vec{0}\}$. This contradicts (21).

Because $\mathcal{U}_{n,B} \neq \emptyset$, the infimum of all $B^* \geq 0$ such that there exists $n^* \in \mathbb{N}$ so that $\mathcal{U}_{B^*,n^*} \neq \emptyset$ is well-defined, formally defined as $B^\#$ as follows.

$$B^\# = \inf\{B^* \geq 0 : \exists n^* \in \mathbb{N} \text{ s.t. } \mathcal{U}_{B^*,n^*} \neq \emptyset\} \tag{23}$$

We note that there exists $n^\# \in \mathbb{N}$ such that $\mathcal{U}_{B^\#,n^\#} \neq \emptyset$, because there are finite number of combinations of operations whose total budget is under B, and $B^\#$ is the minimum budget among the successful combinations (at a non-negative integer vector whose size is $n^\#$). Clearly $B \geq B^\#$. Define

$$\mathcal{R}_B^\infty = \left\{\sum\nolimits_{i=1}^{d_\Delta} \frac{2C'}{C^\infty}\eta_i \cdot \vec{p}_i^{\,\infty} : \forall i \leq d_\Delta, \eta_i \in \{0, 1, \ldots, \left\lfloor \frac{(B - B^\#)C^\infty}{2d_\Delta C'} \right\rfloor\}\right\} \tag{24}$$

where we recall that C' is the constant guaranteed by Claim 7. Intuitively, $\mathcal{R}_B^\infty$ consists of some "grids" in $\frac{2C'}{C^\infty} \cdot \mathbb{L}^\infty \cup \{\vec{0}\}$, which is a subset of $\mathcal{C}_\infty$, because for all $i \leq d_\Delta$, $\vec{p}_i^{\,\infty} \in \mathcal{C}_\infty$.

Step 1.4: Define $\mathcal{R}_{B,n}$ and $\mathcal{R}_{B,n}^{\mathbb{Z}}$. Recall the definition of $B^\#$ from (23) that $\mathcal{U}_{B^\#,n^\#} \neq \emptyset$. Fix $\vec{x}^{\,\#} \in \mathcal{U}_{B^\#,n^\#}$. Let $\vec{x}^{\,@} \in \mathcal{C}_0$ denote an arbitrary interior point of $\mathcal{C}_0$. That is, let $\mathbf{A}_0^+$ denote the remaining rows of $\mathbf{A}_0 = \begin{bmatrix} \mathbf{A}_S \\ \mathbf{A}_T \end{bmatrix}$ after removing the implicit equalities $\mathbf{A}_0^=$, there exists a constant $\epsilon^@ > 0$ such that

$$\mathbf{A}_0^+ \times \left(\vec{x}^{\,@}\right)^\top < -\left(\epsilon^@ \cdot \vec{1}\right)^\top \text{ and } \mathbf{A}_0^= \times \left(\vec{x}^{\,@}\right)^\top = \left(\vec{0}\right)^\top$$

Let $C^@ > 0$ denote an arbitrary constant such that

$$C^@ > \frac{2C'}{C^0} \cdot \frac{\sum_{i=1}^{d_0 - 1} |\mathbf{A}_0^+ \times \left(\vec{p}_i^{\,0}\right)^\top|_\infty}{\epsilon^@} \tag{25}$$

The constraint on $C^@$ in (25) guarantees that for any $\vec{x}^{\,0} \in \mathcal{R}_n^0$, we have $C^@\sqrt{n}\vec{x}^{\,@} + \vec{x}^{\,0} \in \mathcal{C}_0$, which will be formally proved and used in the proof of Claim 9 below.

For any $n \in \mathbb{N}$ and any $\vec{x}^{\,\infty} \in \mathcal{R}_B^\infty$, define

$$\mathcal{R}_{\vec{x}^\infty} = (n - n^\# - C^@\sqrt{n} \cdot (\vec{x}^@ \cdot \vec{1}) - \vec{x}^{\,\infty} \cdot \vec{1}) \cdot \pi^* + \vec{x}^{\,\#} + C^@\sqrt{n}\vec{x}^{\,@} + \mathcal{R}_n^0 + \vec{x}^{\,\infty}$$

It follows that the size of every vector in $\mathcal{R}_{\vec{x}^\infty}$ is n. Let $\mathcal{R}_{B,n} = \bigcup_{\vec{x}^\infty \in \mathcal{R}_B^\infty} \mathcal{R}_{\vec{x}^\infty}$. The following claim states that vectors in $\mathcal{R}_{B,n}$ are $2C'$ away from each other in L_∞, where we recall that C' is the constant guaranteed by Claim 7.

Claim 8 (**Sparsity of $\mathcal{R}_{B,n}$**). *For any pair of vectors $\vec{x}^1, \vec{x}^2 \in \mathcal{R}_{B,n}$ whose $\mathcal{R}_n^0$ or $\mathcal{R}_B^\infty$ components are different, we have $|\vec{x}^1 - \vec{x}^2|_\infty \geq 2C'$.*

Proof. For $j \in \{1, 2\}$, we write

$$\vec{x}^j = \ell^j\pi^* + \vec{x}^{\,\#} + C^@\sqrt{n}\vec{x}^{\,@} + \sum\nolimits_{i=1}^{d_0-1} \frac{2C'}{C^0}\gamma_i^j \cdot \vec{p}_i^{\,0} + \sum\nolimits_{i=1}^{d_\Delta} \frac{2C'}{C^\infty}\eta_i^j \cdot \vec{p}_i^{\,\infty},$$

where ℓ^1 and ℓ^2 guarantee that $\vec{x}^1 \cdot \vec{1} = \vec{x}^2 \cdot \vec{1} = n$. Then,

$$\vec{x}^1 - \vec{x}^2 = \underbrace{(\ell^1 - \ell^2)\pi^*}_{\in \mathcal{P}_0} + \underbrace{\sum\nolimits_{i=1}^{d_0-1} \frac{2C'}{C^0}(\gamma_i^1 - \gamma_i^2) \cdot \vec{p}_i^{\,0}}_{\in \frac{2C'}{C^0} \cdot \mathbb{L}^0 \cup \{\vec{0}\}} + \underbrace{\sum\nolimits_{i=1}^{d_\Delta} \frac{2C'}{C^\infty}(\eta_i^1 - \eta_i^2) \cdot \vec{p}_i^{\,\infty}}_{\in \frac{2C'}{C^\infty} \cdot \mathbb{L}^\infty \cup \{\vec{0}\}}$$

For $j \in \{1, 2\}$, let $\vec{\gamma}^j = (\gamma_1^j, \ldots, \gamma_{d_0-1}^j)$ and $\vec{\eta}^j = (\eta_1^j, \ldots, \eta_{d_\Delta}^j)$. Let $\vec{\gamma}^\Delta = \vec{\gamma}^1 - \vec{\gamma}^2$ and $\vec{\eta}^\Delta = \vec{\eta}^1 - \vec{\eta}^2$. Claim 8 is proved in the following two cases.

- $\vec{\eta}^{\Delta} \neq \vec{0}$. Notice that $\sum_{i=1}^{d_\Delta}(\eta_i^1 - \eta_i^2)\cdot \vec{p}_i^{\,\infty} \in \mathbb{L}^\infty$. Recall that $\pi^* \in \mathcal{C}_0 \subseteq \mathcal{P}_0$, which means that $(\ell^2-\ell^1)\pi^* - \sum_{i=1}^{d_0-1}\frac{2C'}{C^0}(\gamma_i^1-\gamma_i^2)\cdot \vec{p}_i^{\,0} \in \mathcal{P}_0$. According to (22), the L_∞ distance between $\sum_{i=1}^{d_\Delta}(\eta_i^1-\eta_i^2)\cdot \vec{p}_i \in \mathbb{L}^\infty$ and any vector in $\mathcal{P}_0$ is at least C^∞. Therefore,

$$\begin{aligned}|\vec{x}^1 - \vec{x}^2|_\infty =& \left|(\ell^1-\ell^2)\pi^* + \sum_{i=1}^{d_0-1}\frac{2C'}{C^0}(\gamma_i^1-\gamma_i^2)\cdot\vec{p}_i^{\,0} + \sum_{i=1}^{d_\Delta}\frac{2C'}{C^\infty}(\eta_i^1-\eta_i^2)\cdot\vec{p}_i^{\,\infty}\right| \\ =& \frac{2C'}{C^\infty}\cdot\left|\sum_{i=1}^{d_\Delta}(\eta_i^1-\eta_i^2)\cdot\vec{p}_i^{\,\infty} - \frac{C^\infty}{2C'}\left((\ell^2-\ell^1)\pi^* - \sum_{i=1}^{d_0-1}\frac{2C'}{C^0}(\gamma_i^1-\gamma_i^2)\cdot\vec{p}_i^{\,0}\right)\right| \\ \geq& \frac{2C'}{C^\infty}\cdot C^\infty = 2C'\end{aligned}$$

- $\vec{\eta}^{\Delta} = \vec{0}$. In this case we must have $\ell^1=\ell^2$ and $\vec{\gamma}^{\Delta}\neq\vec{0}$. Notice that $\sum_{i=1}^{d_0-1}(\gamma_i^1-\gamma_i^2)\cdot\vec{p}_i^{\,0}\in\mathbb{L}^0$, and recall that the L_∞ norm of any vector in $\mathbb{L}^0$ is at least $C^0>0$, we have

$$\left|\vec{x}^1-\vec{x}^2\right|_\infty = \frac{2C'}{C^0}\cdot\left|\sum_{i=1}^{d_0-1}(\gamma_i^1-\gamma_i^2)\cdot\vec{p}_i^{\,0}\right| \geq \frac{2C'}{C^0}\cdot C^0 = 2C'$$

This completes the proof of Claim 8. □

By Claim 8, $|\mathcal{R}_{B,n}| = \left(1+\left\lfloor\frac{(B-B^\#)C^\infty}{2d_\Delta C'}\right\rfloor\right)^{d_\Delta}\times\left(\lfloor\sqrt{n}\rfloor\right)^{d_0-1}$, which is $\Omega((B+1)^{d_\Delta}\cdot(\sqrt{n})^{d_0-1})$. Next, we prove that for all sufficiently large n, $\mathcal{R}_{B,n}\subseteq\mathcal{H}_{B,n}\cap\mathbb{R}^q_{\geq 0}$.

Claim 9 ($\mathcal{R}_{B,n}\subseteq\mathcal{H}_{B,n}\cap\mathbb{R}^q_{\geq 0}$). *There exists $N\in\mathbb{N}$ that does not depend on B or n, such that when $n\geq N$, $\mathcal{R}_{B,n}\subseteq\mathcal{H}_{B,n}\cap\mathbb{R}^q_{\geq 0}$.*

Proof. For any $\vec{x}\in\mathcal{R}_{B,n}$, we can write

$$\vec{x} = \ell\pi^* + \vec{x}^{\,\#} + C^@\sqrt{n}\vec{x}^{\,@} + \vec{x}^{\,0} + \vec{x}^{\,\infty},$$

where $\vec{x}^{\,0}\in\mathcal{R}_n^0$, $\vec{x}^{\,\infty}\in\mathcal{R}_B^\infty$, and ℓ guarantees that $\vec{x}\cdot\vec{1}=n$. Recall that $\pi^*\in\mathcal{C}_0$ is strictly positive. Therefore, $\vec{x}\geq\vec{0}$ for any sufficiently large n, which means that $\mathcal{R}_{B,n}\subseteq\mathbb{R}^q_{\geq 0}$. The rest of the claim proves $\vec{x}\in\mathcal{H}_{B,n}$ in the following three steps.

(1) $C^@\sqrt{n}\vec{x}^{\,@}+\vec{x}^{\,0}\in\mathcal{C}_0$. Let $\vec{x}'' = C^@\sqrt{n}\vec{x}^{\,@}+\vec{x}^{\,0}$, we prove the following.

(1.1) $\mathbf{A}_0^= \times (\vec{x}'')^\top = (\vec{0})^\top$. Recall that $\vec{x}^{\,@}$ is an interior point of $\mathcal{C}_0\subseteq\mathcal{H}_{\mathrm{S},\leq 0}$, which means that $\mathbf{A}_0^=\times(\vec{x}^{\,@})^\top = (\vec{0})^\top$. Also recall that $\mathcal{R}_n^0\subseteq\mathcal{P}^0$, which means that $\mathbf{A}_0^=\times(\vec{x}^{\,0})^\top = (\vec{0})^\top$. Therefore,

$$\mathbf{A}_0^=\times(\vec{x}'')^\top = C^@\sqrt{n}\mathbf{A}_0^=\times\left(\vec{x}^{\,@}\right)^\top + \mathbf{A}_0^=\times\left(\vec{x}^{\,0}\right)^\top = \left(\vec{0}\right)^\top$$

(1.2) $\mathbf{A}_0^+\times(\vec{x}'')^\top < (\vec{0})^\top$. Let $\vec{x}^{\,0} = \sum_{i=1}^{d_0-1}\frac{2C'}{C^0}\gamma_i^j\cdot\vec{p}_i^{\,0}$. Recall that $\mathbf{A}_0^+\times(\vec{x}^{\,@})^\top < \left(-\epsilon^@\cdot\vec{1}\right)^\top$, and recall the lower bound of $C^@$ from (25). We have

$$\mathbf{A}_0^+\times(\vec{x}'')^\top = C^@\sqrt{n}\mathbf{A}_0^+\times\left(\vec{x}^{\,@}\right)^\top + \sum_{i=1}^{d_0-1}\frac{2C'}{C^0}\gamma_i^j\cdot\mathbf{A}_0^+\times\left(\vec{p}_i^{\,0}\right)^\top < \left(\vec{0}\right)^\top$$

(2) $\vec{x}\in\mathcal{H}_{\mathrm{S}}$. Recall that $\pi^*\in\mathcal{C}_0$, $\vec{x}^{\,\#}\in\mathcal{U}_{B^\#,n^\#}\subseteq\mathcal{H}_{\mathrm{S}}$, and $\vec{x}^{\,\infty}\in\mathcal{C}_\infty\subseteq\mathcal{H}_{\mathrm{S},\leq 0}$. Therefore,

$$\begin{aligned}\mathbf{A}_{\mathrm{S}}\times(\vec{x})^\top =& \ell\mathbf{A}_{\mathrm{S}}\times\underbrace{(\pi^*)}_{\text{in } \mathcal{C}_0}{}^\top + \mathbf{A}_{\mathrm{S}}\times\underbrace{(\vec{x}^{\,\#})}_{\text{in } \mathcal{H}_{\mathrm{S}}}{}^\top + \mathbf{A}_{\mathrm{S}}\times\underbrace{(C^@\sqrt{n}\vec{x}^{\,@}+\vec{x}^{\,0})}_{\text{in } \mathcal{C}_0}{}^\top + \mathbf{A}_{\mathrm{S}}\times\underbrace{(\vec{x}^{\,\infty})}_{\text{in } \mathcal{H}_{\mathrm{S},\leq 0}}{}^\top \\ \leq& \left(\vec{0}\right)^\top + (\mathbf{b}_{\mathrm{S}})^\top + \left(\vec{0}\right)^\top + \left(\vec{0}\right)^\top = (\mathbf{b}_{\mathrm{S}})^\top\end{aligned}$$

This proves $\vec{x}\in\mathcal{H}_{\mathrm{S}}$.

(3) $\vec{x}$ is B-manipulable. Because $\vec{x}^{\,\#}\in\mathcal{U}_{B^\#,n^\#}$, there exists $\vec{o}^{\,\#}\in\mathbb{Z}_{\geq 0}^{|\mathbb{O}|}$ such that $\vec{c}\cdot\vec{o}^{\,\#}\leq B^\#$ and $\vec{x}^{\,\#}+\vec{o}^{\,\#}\times\mathbf{O}\in\mathcal{H}_{\mathrm{T}}$. Let $\vec{x}^{\,\infty} = \sum_{i=1}^{d_\Delta}\frac{2C'}{C^\infty}\eta_i\cdot\vec{p}_i^{\,\infty}$. Also recall that for all $i\leq d_\Delta$, $\vec{p}_i^{\,\infty} = \vec{y}_i^{\,\infty} - \vec{o}_i^{\,\infty}\times\mathbf{O}$, where $\vec{y}_i^{\,\infty}\in\mathcal{H}_{\mathrm{T},\leq 0}$, $\vec{o}_i^{\,\infty}\geq\vec{0}$, and $\vec{c}\cdot\vec{o}_i^{\,\infty}=1$. Let $\vec{o}^{\,*} = \vec{o}^{\,\#}+\sum_{i=1}^{d_\Delta}\frac{2C'}{C^\infty}\eta_i\cdot\vec{o}_i^{\,\infty}$. We prove that $\vec{c}\cdot\vec{o}^{\,*}\leq B$ as follows.

$$\begin{aligned}\vec{c}\cdot\vec{o}^{\,*} &= \vec{c}\cdot\vec{o}^{\,\#} + \sum_{i=1}^{d_\Delta}\frac{2C'}{C^\infty}\eta_i\cdot\vec{c}\cdot\vec{o}_i^{\,\infty} \\ &\leq B^\# + \sum_{i=1}^{d_\Delta}\frac{2C'}{C^\infty}\cdot\frac{(B-B^\#)C^\infty}{2d_\Delta C'} = B^\# + B - B^\# = B\end{aligned}$$

Then,

$$
\begin{aligned}
&\mathbf{A}_{\mathrm{T}} \times \left(\vec{x} + \vec{o}^* \times \mathbf{O}\right)^\top \\
=&\ell \mathbf{A}_{\mathrm{T}} \times \underbrace{(\pi^*)}_{\text{in } C_0}{}^\top + \mathbf{A}_{\mathrm{T}} \times \underbrace{(\vec{x}^{\#} + \vec{o}^{\#} \times \mathbf{O})}_{\text{in } \mathcal{H}_{\mathrm{T}}}{}^\top + \mathbf{A}_{\mathrm{T}} \times \underbrace{(C^{@}\sqrt{n}\vec{x}^{@} + \vec{x}^{0})}_{\text{in } C_0}{}^\top \\
&\qquad + \sum\nolimits_{i=1}^{d_\Delta} \frac{2C'}{C^\infty}\eta_i \cdot \mathbf{A}_{\mathrm{T}} \times \underbrace{(\vec{p}_i^{\,\infty} + \vec{o}_i^{\,\infty} \times \mathbf{O})}_{=\vec{y}^{\,i} \in \mathcal{H}_{\mathrm{T},\leq 0}}{}^\top \\
\leq& \left(\vec{0}\right)^\top + (\mathbf{b}_{\mathrm{T}})^\top + \left(\vec{0}\right)^\top + \left(\vec{0}\right)^\top = (\mathbf{b}_{\mathrm{T}})^\top
\end{aligned}
$$

This proves that $\vec{x} + \vec{o}^* \times \mathbf{O} \in \mathcal{H}_{\mathrm{T}}$ and completes the proof of Claim 9. □

By Claim 7, for every $\vec{x} \in \mathcal{R}_{B,n}$, there exists $\vec{x}' \in \mathcal{U}_{n,B}$ that is no more than C' away from $\vec{x}$ in L_∞. Because vectors in $\mathcal{R}_{B,n}$ are at least $2C'$ away from each other, their corresponding vectors in $\mathcal{U}_{n,B}$ are different, denoted by $\mathcal{R}_{B,n}^{\mathbb{Z}}$ and is formally defined as follows.

Definition 18 ($\mathcal{R}_{B,n}^{\mathbb{Z}}$). *Let $\mathcal{R}_{B,n}^{\mathbb{Z}}$ denote the integer vectors corresponding to the vectors in $\mathcal{R}_{B,n}$ guaranteed by Claim 7.*

It follows that $|\mathcal{R}_{B,n}^{\mathbb{Z}}| = |\mathcal{R}_{B,n}| = \Omega\left((B+1)^{d_\Delta} \cdot \left(\sqrt{n}\right)^{d_0-1}\right)$. This completes Step 1 for poly lower bound.

Step 2 for poly lower bound: Define $\vec{\pi}^\circ$ for any $\pi^* \in \mathrm{CH}(\Pi)$. In this step, we define $\vec{\pi}^\circ = (\pi_1^\circ, \ldots, \pi_n^\circ) \in \Pi^n$ such that $\sum_{j=1}^n \pi_j^\circ$ is $O(\sqrt{n})$ away from $n\pi^*$ in a way that is similar to [77].

More precisely, because $\pi^* \in \mathrm{CH}(\Pi)$, by Carathéodory's theorem for convex/conic hulls (see e.g., [42, p. 257]), we can write π^* as the convex combination of $1 \leq t \leq q$ distributions in Π, i.e.,

$$\pi^* = \sum\nolimits_{i=1}^t \alpha_i \pi_i^*, \text{ where } \sum\nolimits_{i=1}^t \alpha_i = 1 \text{ and } \pi_i^* \in \Pi$$

We note that $\pi^* \geq \epsilon \cdot \vec{1}$, because Π is strictly positive (by ϵ). For each $i \leq t-1$, let $\vec{\pi}_i^\ell$ denote the vector of $\beta_i = \lfloor \ell\alpha_i \rfloor$ copies of π_i^*. Let $\vec{\pi}_k^\ell$ denote the vector of $\beta_t = n - \sum_{i=1}^{t-1}\beta_i$ copies of π_t^*. It follows that for any $i \leq t-1$, $|\beta_i - \ell\alpha_i| \leq 1$, and $|\beta_t - \ell\alpha_t| \leq t + (\vec{y}^n \cdot \vec{1} - \ell\pi^* \cdot \vec{1}) = O(\sqrt{\ell}) = O(\sqrt{n})$.

Let $\vec{\pi}^\circ = (\vec{\pi}_1^\ell, \ldots, \vec{\pi}_t^\ell)$, or equivalently

$$\vec{\pi}^\circ = (\underbrace{\pi_1^*, \ldots, \pi_1^*}_{\beta_1}, \underbrace{\pi_2^*, \ldots, \pi_2^*}_{\beta_2}, \ldots, \underbrace{\pi_t^*, \ldots, \pi_t^*}_{\beta_t})$$

Step 3 for poly lower bound: Lower-bound $\Pr_{P\sim\vec{\pi}}(\vec{X}_{\vec{\pi}} \in \mathcal{H}_B)$.

Claim 10. *For any PMV-instability setting $\mathcal{S}$, any strictly positive set of distributions Π (by $\epsilon > 0$), and any $\alpha > 0$, there exist $C_{\mathcal{S}} > 0$ and $N > 0$ such that for any $n \geq N$, any $0 \leq B \leq \sqrt{n}$, and any $\vec{\pi} \in \Pi^n$ such that the L_∞ distance between $\sum_{j=1}^n \pi_j$ and C_0 is no more than $\alpha\sqrt{n}$,*

$$\Pr(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}) \geq C_{\mathcal{S}} \cdot (B+1)^{d_\Delta} \cdot \left(\frac{1}{\sqrt{n}}\right)^{q-d_0}$$

Proof. Let $\pi^* \in C_0$ denote an arbitrary vector such that $|\sum_{j=1}^n \pi_j - n\pi^*|_\infty < 2\alpha\sqrt{n}$. Let $\mathcal{R}_{B,n}^{\mathbb{Z}}$ denote the set of integer vectors for π^* (Definition 18 in Step 1.4 above). Because $B = O(\sqrt{n})$, each vector in $\mathcal{R}_{B,n}^{\mathbb{Z}}$ is $O(\sqrt{n})$ away from $n\pi^*$, which is $O(\sqrt{n})$ away from $\sum_{j=1}^n \pi_j$. By the point-wise concentration bound ([77, Lemma 1]), for every $\vec{x} \in \mathcal{R}_{B,n}^{\mathbb{Z}}$, we have $\Pr_{P\sim\vec{\pi}}(\mathrm{Hist}(P) = \vec{x}) = \Omega((\sqrt{n})^{1-q})$. Therefore,

$$
\begin{aligned}
&\Pr(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}) \geq \Pr_{P\sim\vec{\pi}}(\mathrm{Hist}(P) \in \mathcal{R}_{B,n}^{\mathbb{Z}}) \geq |\mathcal{R}_{B,n}^{\mathbb{Z}}| \times \Omega\left((\sqrt{n})^{1-q}\right) \\
=&\Omega\left((B+1)^{d_\Delta} \cdot (\sqrt{n})^{d_0-1}\right) \times \Omega\left((\sqrt{n})^{1-q}\right) = \Omega\left((B+1)^{d_\Delta} \cdot (\frac{1}{\sqrt{n}})^{q-d_0}\right)
\end{aligned}
$$

This completes proof of Claim 10. □

The lower bound for PT-$\Theta(\sqrt{n})$-sup follows after applying Claim 10 to any $\pi^* \in \mathrm{CH}(\Pi) \cap C_0$ and $\vec{\pi}^\circ$ defined in Step 2 for poly lower bound above.

Proof for the phase transition at $\Theta(n)$ case of sup (PT-$\Theta(n)$-sup for short).

$B \leq C_2 n$. Let $c^* = \frac{1}{2}(C_2 + B_{\mathrm{CH}(\Pi)})$. We first show that $n \cdot \mathrm{CH}(\Pi)$ is $\Omega(n)$ away from C_{nc^*}, which is equivalent to $\mathrm{CH}(\Pi)$ being $\Omega(1)$ away from C_{c^*}. Due to the minimality of $B_{\mathrm{CH}(\Pi)}$, we have $\mathrm{CH}(\Pi) \cap C_{c^*} = \emptyset$. Notice that $\mathrm{CH}(\Pi)$ is convex and compact and C_{c^*} is convex. By the strict separating hyperplane theorem, the distance between $\text{CH}(\Pi)$ and $\mathcal{C}_{c^*}$ is strictly positive, denoted by c'. Let $c_2 > 0$ denote any fixed constant that is smaller than c'.

Next, we prove that $n \cdot \text{CH}(\Pi)$ is $\Omega(n)$ away from $\mathcal{H}_B$. To this end, we prove the following claim, which states that for any sufficiently large $B' \geq 0$, $\mathcal{H}_{B'}$ is $O(1)$ away from $\mathcal{C}_{B'+O(1)}$.

Claim 11. *Given a PMV-instability setting $\mathcal{S}$, there exists a constant C such that for any $B' \geq C$, any n, and any $\vec{x} \in \mathcal{H}_{B'}$, there exists $\vec{x}' \in \mathcal{C}_{B'+C}$ such that $|\vec{x} - \vec{x}'|_\infty \leq C$.*

Proof. The proof is done by analyzing feasible solutions to the following two linear programs $\text{LP}_{\mathcal{H}}$ and LP_{Cone}, whose variables are $\vec{x}$ and $\vec{o}$.

$\text{LP}_{\mathcal{H}}$	LP_{Cone}
$\begin{aligned} \max \quad & 0 \\ \text{s.t.} \quad & \mathbf{A}_S \times (\vec{x})^\top \leq (\mathbf{b}_S)^\top \\ & \mathbf{A}_T \times (\vec{x} + \vec{o} \times \mathbf{O})^\top \leq (\mathbf{b}_T)^\top \\ & -\vec{o} \leq \vec{0} \end{aligned}$	$\begin{aligned} \max \quad & 0 \\ \text{s.t.} \quad & \mathbf{A}_S \times (\vec{x})^\top \leq \left(\vec{0}\right)^\top \\ & \mathbf{A}_T \times (\vec{x} + \vec{o} \times \mathbf{O})^\top \leq \left(\vec{0}\right)^\top \\ & -\vec{o} \leq \vec{0} \end{aligned}$

Because $\vec{x} \in \mathcal{H}_{B'}$, there exists $\vec{o} \geq \vec{0}$ such that $\vec{x} + \vec{o} \times \mathbf{O} \in \mathcal{H}_T$ and $\vec{c} \cdot \vec{o} \leq B'$. Therefore, $(\vec{x}, \vec{w})$ is a feasible solution to $\text{LP}_{\mathcal{H}}$. Notice that LP_{Cone} is feasible (for example $\vec{0}$ is a feasible solution) and $\text{LP}_{\mathcal{H}}$ and LP_{Cone} only differ on the right hand side of the inequalities. Therefore, due to [15, Theorem 5 (i)], there exists a feasible solution $(\vec{x}', \vec{w}')$ to LP_{Cone} that is no more than $q\Delta \max\{|\mathbf{b}_S - \vec{0}|_\infty, |\mathbf{b}_T - \vec{0}|_\infty\} = O(1)$ away from $(\vec{x}, \vec{w})$ in L_∞, where Δ is the maximum absolute value of determinants of square sub-matrices of the left hand side of $\text{LP}_{\mathcal{H}}$ and LP_{Cone}, i.e., $\begin{bmatrix} \mathbf{A}_S & 0 \\ \mathbf{A}_T & \mathbf{A}_T \times (\mathbf{O})^\top \\ 0 & -\mathbb{I} \end{bmatrix}$. This implies that

$$\vec{c} \cdot \vec{o}' \leq \vec{c} \cdot \vec{w} + (\vec{c} \cdot \vec{1}) q\Delta \max\{|\mathbf{b}_S|_\infty, |\mathbf{b}_T|_\infty\}$$

The claim follows after letting $C = (\vec{c} \cdot \vec{1}) q\Delta \max\{|\mathbf{b}_S|_\infty, |\mathbf{b}_T|_\infty\}$. □

As proved above, $n \cdot \text{CH}(\Pi)$ is $\Omega(n)$ away from $\mathcal{C}_{nc^*}$ for any sufficiently large n. Notice that for any sufficiently large n, we have $(c^* - C_2)n > C$, where C is the constant guaranteed by Claim 11. Therefore, by Claim 11, every vector $\vec{x}$ in $\mathcal{H}_B$ is $O(1)$ away from a vector $\vec{x}'$ in $\mathcal{C}_{B+O(1)} \subseteq \mathcal{C}_{nc^*}$. Recall from above that $n \cdot \text{CH}(\Pi)$ is $\Omega(n)$ away from $\mathcal{C}_{nc^*}$. Therefore, $n \cdot \text{CH}(\Pi)$ is $\Omega(n)$ away from $\mathcal{H}_B$.

Finally, for any $\vec{\pi} \in \Pi^n$, let $\pi' = \frac{1}{n}\sum_{j=1}^n \pi_j$. Because $\pi' \in \text{CH}(\Pi)$, the L_∞ distance between π', which is the mean vector of $\vec{X}_{\vec{\pi}}$, and $\mathcal{H}_B$ is $\Omega(n)$. The (exponential) upper bound for the $B \leq C_2 n$ case is proved by a straightforward application of Hoeffding's inequality and the union bound to all q dimensions as in the proof of the exponential case of sup. The exponential lower bound trivially holds.

$B \geq C_3 n$. The $O\left(\left(\frac{1}{\sqrt{n}}\right)^{q-d_\infty}\right)$ upper bound of this case follows after Claim 6. At a high level, the proof of the $\Omega\left(\left(\frac{1}{\sqrt{n}}\right)^{q-d_\infty}\right)$ lower bound is similar to the proof of the polynomial lower bound of PT-$\Theta(\sqrt{n})$-sup. The main difference is that the condition is weaker now $(\text{CH}(\Pi) \cap \mathcal{C}_\infty \neq \emptyset$ compared to $\mathcal{C}_0 \cap \text{CH}(\Pi) \neq \emptyset)$. Therefore, we will identify two different sets $\mathcal{R}^+_{B,n} \subseteq \mathcal{H}_{B,n}$ and $\mathcal{R}^{\mathbb{Z}+}_{B,n} \subseteq \mathcal{U}_{n,B}$ for any strictly positive $\pi^+ \in \text{Cone}_{B^+}$ with $\pi^+ \cdot \vec{1} = 1$, where B^+ is a fixed number such that $C_3 > B^+ > B_{\text{CH}(\Pi)}$. It follows that $\pi^+ \in \mathcal{H}_{S,\leq 0}$ and we can write $\pi^+ = \vec{y}^+ - \vec{o}^+ \times \mathbf{O}$, where $\vec{y}^+ \in \mathcal{H}_{T,\leq 0}$, $\vec{o}^+ \geq \vec{0}$, and $\vec{c} \cdot \vec{o}^+ \leq B^+$.

In the following procedure, we define a basis $\mathbf{B}^+$ of $\mathcal{P}_\infty$ that is similar to B^∞. Let $\{\vec{p}_1^+, \ldots, \vec{p}_{d_\infty}^+\}$ denote a set of d_∞ linearly independent vectors in $\mathcal{C}_\infty \subseteq \mathcal{P}_\infty$. W.l.o.g. suppose for every $j \leq d_\infty$, $\vec{p}_j^+ \cdot \vec{1} \in \{-1, 0, 1\}$—otherwise we divide $\vec{p}_j^+$ by $|\vec{p}_j^+ \cdot \vec{1}|$. For every $j \leq d_\infty$, let $\vec{p}_j^+ = \vec{y}_j^+ - \vec{o}_j^+ \times \mathbf{O}$, where $\vec{y}_j^+ \in \mathcal{H}_{T,\leq 0}$ and $\vec{o}_j^+ \geq \vec{0}$. For convenience, let $\vec{p}_0^+ \triangleq \pi^+$.

Procedure. Start with $\mathbf{B}^+ = \{\vec{p}_0^+\}$. For every $1 \leq j \leq d_\infty$, we add $\vec{p}_j^+$ to $\mathbf{B}^+$ if and only if it is linearly independent of existing vectors in $\mathbf{B}^+$. At the end of the procedure, we have $|\mathbf{B}^+| = d_\infty$. W.l.o.g., let $\mathbf{B}^+ = \{\vec{p}_0^+, \vec{p}_1^+, \ldots, \vec{p}_{d_\infty - 1}^+\}$.

Next, we define $\mathbb{L}^+$, C^+, and $\mathcal{R}_n^+$ that are similar to $\mathbb{L}^0$, C^0, and $\mathcal{R}_n^0$ in Step 1.2 of the proof of the polynomial lower bound of PT-$\Theta(\sqrt{n})$-sup, respectively.

Let $\mathbb{L}^+$ denote the lattice generated by $\mathbf{B}^+$ excluding $\vec{0}$. That is,

$$\mathbb{L}^+ = \left\{\sum_{i=0}^{d_\infty - 1} \lambda_i \cdot \vec{p}_i^+ : \forall 0 \leq i \leq d_\infty - 1, \lambda_i \in \mathbb{Z} \text{ and } \exists 0 \leq i \leq d_\infty - 1 \text{ s.t. } \lambda_i \neq 0\right\}$$

Let C^+ denote the minimum L_∞ norm of vectors in $\mathbb{L}^+$. That is,

$$C^+ = \inf\left\{|\vec{x}|_\infty : \vec{x} \in \hat{\mathbb{L}}\right\}$$

Following a similar argument that proves $C^0 > 0$, we have $C^+ > 0$. Then, define

$$\mathcal{R}_n^+ = \left\{ \sum_{i=1}^{d_\infty - 1} \left\lceil \frac{2C'}{C^+} \right\rceil \cdot \lambda_i \cdot \vec{p}_i^{\,+} : \forall 1 \le i \le d_\infty - 1, \lambda_i \in \{0, 1, \ldots, \lfloor \sqrt{n} \rfloor\} \right\}$$

Notice that vectors in $\mathcal{R}_n^+$ do not have $\vec{p}_0 = \pi^+$ components. Recall that $\vec{x}^\# \in \mathcal{U}_{B^\#, n^\#}$, which means that $\vec{x}^\# \cdot \vec{1} = n^\#$. For any $n \in \mathbb{N}$, define

$$\mathcal{R}_{B,n}^+ = \left\{ (n - n^\# - \vec{x}^{\,+} \cdot \vec{1}) \cdot \pi^+ + \vec{x}^{\,\#} + \vec{x}^{\,+} : \vec{x}^{\,+} \in \mathcal{R}_n^+ \right\}$$

Like Claim 8, in the following claim we prove that vectors in $\mathcal{R}_{B,n}^+$ are at least $2C'$ from each other in L_∞.

Claim 12 (**Sparsity of $\mathcal{R}_{B,n}^+$**). *For any pair of vectors $\vec{x}^1, \vec{x}^2 \in \mathcal{R}_{B,n}^+$ whose $\mathcal{R}_n^+$ components are different, we have $|\vec{x}^1 - \vec{x}^2|_\infty \ge 2C'$.*

Proof. For $j \in \{1, 2\}$, we write

$$\vec{x}^j = \ell^j \cdot \pi^+ + \vec{x}^{\,\#} + \sum_{i=1}^{d_\infty - 1} \left\lceil \frac{2C'}{C^+} \right\rceil \cdot \lambda_i^j \cdot \vec{p}_i^{\,+},$$

where ℓ^1 and ℓ^2 guarantee that $\vec{x}^1 \cdot \vec{1} = \vec{x}^2 \cdot \vec{1} = n$. Then, because $(\vec{x}^1 - \vec{x}^2) \cdot \vec{1} = 0$ and $\pi^+ \cdot \vec{1} = 1$, we have

$$\ell^1 - \ell^2 = \sum_{i=1}^{d_\infty - 1} \left\lceil \frac{2C'}{C^+} \right\rceil (\lambda_i^2 - \lambda_i^1) \cdot (\vec{p}_i^{\,+} \cdot \vec{1})$$

Therefore,

$$\begin{aligned}
\vec{x}^1 - \vec{x}^2 =& (\ell^1 - \ell^2)\pi^+ + \sum_{i=1}^{d_\infty - 1} \left\lceil \frac{2C'}{C^+} \right\rceil (\lambda_i^1 - \lambda_i^2) \cdot \vec{p}_i^{\,+} \\
=& \sum_{i=1}^{d_\infty - 1} \left\lceil \frac{2C'}{C^+} \right\rceil (\lambda_i^2 - \lambda_i^1) \cdot (\vec{p}_i^{\,+} \cdot \vec{1}) \cdot \pi^+ + \sum_{i=1}^{d_\infty - 1} \left\lceil \frac{2C'}{C^+} \right\rceil (\lambda_i^1 - \lambda_i^2) \cdot \vec{p}_i^{\,+} \\
=& \left\lceil \frac{2C'}{C^+} \right\rceil \left(\sum_{i=1}^{d_\infty - 1} (\lambda_i^2 - \lambda_i^1) \cdot (\vec{p}_i^{\,+} \cdot \vec{1}) \cdot \pi^+ + \sum_{i=1}^{d_\infty - 1} (\lambda_i^1 - \lambda_i^2) \cdot \vec{p}_i^{\,+} \right)
\end{aligned}$$

Recall that for all $i \le d_\infty - 1$, $\vec{p}_i^{\,+} \cdot \vec{1}$ is an integer, which means that $\sum_{i=1}^{d_\infty - 1} (\lambda_i^2 - \lambda_i^1) \cdot (\vec{p}_i^{\,+} \cdot \vec{1})$ is an integer. Also because the $\mathcal{R}_n^+$ components of $\vec{x}^1$ and $\vec{x}^2$ are different, there exists $i \le d_\infty - 1$ such that $\lambda_i^1 - \lambda_2^2 \ne 0$. Therefore,

$$\sum_{i=1}^{d_\infty - 1} (\lambda_i^2 - \lambda_i^1) \cdot (\vec{p}_i^{\,+} \cdot \vec{1}) \cdot \pi^+ + \sum_{i=1}^{d_\infty - 1} (\lambda_i^1 - \lambda_i^2) \cdot \vec{p}_i^{\,+} \in \mathbb{L}^+$$

Therefore,

$$\begin{aligned}
\left| \vec{x}^1 - \vec{x}^2 \right|_\infty =& (\ell^1 - \ell^2)\pi^+ + \sum_{i=1}^{d_\infty - 1} \left\lceil \frac{2C'}{C^+} \right\rceil (\lambda_i^1 - \lambda_i^2) \cdot \vec{p}_i^{\,+} \\
=& \sum_{i=1}^{d_\infty - 1} \left\lceil \frac{2C'}{C^+} \right\rceil (\lambda_i^2 - \lambda_i^1) \cdot (\vec{p}_i^{\,+} \cdot \vec{1}) \cdot \pi^+ + \sum_{i=1}^{d_\infty - 1} \left\lceil \frac{2C'}{C^+} \right\rceil (\lambda_i^1 - \lambda_i^2) \cdot \vec{p}_i^{\,+} \\
=& \left\lceil \frac{2C'}{C^+} \right\rceil \left| \sum_{i=1}^{d_\infty - 1} (\lambda_i^2 - \lambda_i^1) \cdot (\vec{p}_i^{\,+} \cdot \vec{1}) \cdot \pi^+ + \sum_{i=1}^{d_\infty - 1} (\lambda_i^1 - \lambda_i^2) \cdot \vec{p}_i^{\,+} \right|_\infty \\
\ge& \left\lceil \frac{2C'}{C^+} \right\rceil \cdot C^+ = 2C'
\end{aligned}$$

This completes the proof of Claim 12. □

Then, we prove the counterpart of Claim 9 for $\mathcal{R}_{B,n}^+$ in the following claim.

Claim 13 (**$\mathcal{R}_{B,n}^+ \subseteq \mathcal{H}_{B,n} \cap \mathbb{R}_{\ge 0}^q$**). *There exists $N \in \mathbb{N}$ that does not depend on B or n, such that when $n \ge N$, $\mathcal{R}_{B,n}^+ \subseteq \mathcal{H}_{B,n} \cap \mathbb{R}_{\ge 0}^q$.*

Proof. Let $\vec{x} = (n - n^\# - \vec{x}^{\,+} \cdot \vec{1}) \cdot \pi^+ + \vec{x}^{\,\#} + \vec{x}^{\,+}$ denote any vector in $\mathcal{R}_{B,n}^+$, where $\vec{x}^{\,+} = \sum_{i=1}^{d_\infty - 1} \left\lceil \frac{2C'}{C^+} \right\rceil \cdot \lambda_i \cdot \vec{p}_i^{\,+}$. Clearly $\mathcal{R}_{B,n}^+ \subseteq \mathbb{R}_{\ge 0}^q$ for sufficiently large n, because π^+ is strictly positive. It is not hard to verify that $\vec{x} \in \mathcal{H}_{\mathrm{S}}$, because $\pi^+ \in \mathcal{H}_{\mathrm{S}, \le 0}$, $\vec{x}^{\,\#} \in \mathcal{H}_{\mathrm{S}}$, and $\vec{x}^{\,+} \in \mathcal{H}_{\mathrm{S}, \le 0}$. Let

$$\vec{o} = (n - n^\# - \vec{x}^{\,+} \cdot \vec{1}) \cdot \vec{o}^{\,+} + \vec{o}^{\,\#} + \sum_{i=1}^{d_\infty - 1} \left\lceil \frac{2C'}{C^+} \right\rceil \cdot \lambda_i \cdot \vec{o}_i^{\,+}$$

It follows that $\vec{c} \cdot \vec{w} \le B^+ n + B^\# + O(\sqrt{n})$, which is smaller than $B \ge C_3 n$ for any sufficiently large n, because $C_3 > B^+$. Then,

$$\vec{x} + \vec{w} \times \mathbf{O} = \left(n - n^\# - \vec{x}^{\,+} \cdot \vec{1} \right) \cdot \underbrace{\vec{y}^{\,+}}_{\text{in } \mathcal{H}_{\mathrm{T}, \le 0}} + \underbrace{\vec{y}^{\,\#}}_{\text{in } \mathcal{H}_{\mathrm{T}}} + \sum_{i=1}^{d_\infty - 1} \left\lceil \frac{2C'}{C^+} \right\rceil \cdot \lambda_i \cdot \underbrace{\vec{y}_i}_{\text{in } \mathcal{H}_{\mathrm{T}, \le 0}} \in \mathcal{H}_{\mathrm{T}}$$

Therefore, $\mathcal{R}_{B,n}^+ \subseteq \mathcal{H}_{B,n}$. This completes the proof of Claim 13. □

Let $\mathcal{R}^{\mathbb{Z}+}_{B,n} \subseteq \mathcal{U}_{n,B}$ denote the integer vectors corresponding to vectors in $\mathcal{R}^{+}_{B,n}$ guaranteed by Claim 7. Then, $|\mathcal{R}^{\mathbb{Z}+}_{B,n}| = \Omega((\sqrt{n})^{d_\infty - 1})$. The $\Omega\left((\frac{1}{\sqrt{n}})^{q-d_\infty}\right)$ lower bound follows after similar steps as Step 2 and Step 3 for the polynomial lower bound of PT-$\Theta(\sqrt{n})$-sup. Specifically, we have the following counterpart to Claim 10 with a similar proof.

Claim 14. *For any PMV-instability setting $\mathcal{S}$, any strictly positive set of distributions Π such that $B^-_{CH(\Pi)} < \infty$, and any $C^-_3 > B^-_{CH(\Pi)}$, there exist $C_{\mathcal{S}} > 0$ and $N > 0$ such that for any $n \geq N$, any $B \geq C^-_3 n$, and any $\vec{\pi} \in \Pi^n$,*

$$\Pr(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}) \geq C_{\mathcal{S}} \cdot \left(\frac{1}{\sqrt{n}}\right)^{q-d_\infty}$$

Proof for inf. Like the sup part, we call the four cases in inf part the 0 case, the exponential case, the phase transition at $\Theta(\sqrt{n})$ case, and the phase transition at $\Theta(n)$ case.

The 0 case is straightforward. To prove the exponential upper bound, we need to prove that there exists $\vec{\pi} \in \Pi^n$ such that $\Pr(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}) = \exp(-\Omega(n))$. Let π' denote an arbitrary vector in $\text{CH}(\Pi) \setminus \mathcal{C}_\infty$. Because $\mathcal{C}_\infty$ is a closed cone, the distance between π' and $\mathcal{C}_\infty$ is strictly positive, which means that the distance between $n \cdot \pi'$ and $\mathcal{C}_\infty$ is $\Omega(n)$. Then, let $\vec{\pi} \in \Pi^n$ denote an arbitrary vector such that $\frac{1}{n}\sum_{j=1}^n \pi_j$ is $O(1)$ from π' in L_∞. Therefore, $\sum_{j=1}^n \pi_j$, which is the mean vector of $\vec{X}_{\vec{\pi}}$, is $\Theta(n)$ away from $\mathcal{C}_\infty \supseteq \mathcal{C}_B$. Then, similar to the proof for the exponential case of sup, by Claim 3, $\sum_{j=1}^n \pi_j$ is $\Theta(n)$ away from $\mathcal{H}_B \supseteq \mathcal{U}_{n,B}$. The exponential upper bound of inf then follows after Hoeffding's inequality and the union bound. The exponential lower bound trivially holds.

Proof for the phase transition at $\Theta(\sqrt{n})$ case of inf. The (polynomial) upper bound follows after the upper bound of the phase transition at $\Theta(\sqrt{n})$ case of sup. The polynomial lower bound follows after applying Claim 10 to every $\vec{\pi} = (\pi_1, \ldots, \pi_n) \in \Pi^n$ and letting $\pi^* = \frac{1}{n}\sum_{j=1}^n \pi_j \in \mathcal{C}_0$.

Proof for the phase transition at $\Theta(n)$ case of inf. When $B \leq C^-_2 n$, the (exponential) lower bound is straightforward. The proof for the (exponential) upper bound is similar to the proof of the $B \leq C_2 n$ case of PT-$\Theta(n)$-sup. More precisely, let $c^* = \frac{1}{2}(C^-_2 + B^-_{\text{CH}(\Pi)})$. According to the minimality of $B^-_{\text{CH}(\Pi)}$, we have $\text{CH}(\Pi) \not\subseteq \mathcal{C}_{c^*}$. Let $\pi' \in \text{CH}(\Pi) \setminus \mathcal{C}_{c^*}$, which means that π' is $\Omega(1)$ away from $\mathcal{C}_{c^*}$, because $\mathcal{C}_{c^*}$ is a closed set. Let $\vec{\pi} \in \Pi^n$ denote an arbitrary vector such that $\sum_{j=1}^n \pi_j$ is $O(1)$ from $n \cdot \pi'$ in L_∞. Then, $\sum_{j=1}^n \pi_j$ is $\Omega(n)$ away from $\mathcal{C}_{c^*}$. By Claim 11, $\sum_{j=1}^n \pi_j$, which is the mean vector of $\vec{X}_{\vec{\pi}}$, is $\Omega(n)$ away from $\mathcal{H}_B$ in L_∞. The exponential upper bound follows after Hoeffding's inequality and the union bound.

When $B \geq C^-_3 n$, the polynomial upper bound follows after the sup case. To prove the polynomial lower bound, notice that $C^-_3 > B^-_{\text{CH}(\Pi)} \geq B_{\text{CH}(\Pi)}$. Let B^+ be any number such that $B^-_{\text{CH}(\Pi)} < B^+ < C^-_3$. This means that $\text{CH}(\Pi) \subseteq \mathcal{C}_{B^+}$. Because $B^-_{\text{CH}(\Pi)} \geq B_{\text{CH}(\Pi)}$, we have $B^+ > B_{\text{CH}(\Pi)}$. The proof for the lower bound is similar to the proof for the lower bound in the phase transition at $\Theta(n)$ case of inf: notice that in the proof of the $B \geq C_3 n$ case of PT-$\Theta(n)$-sup, the proof works for any $\pi^+ \in \text{CH}(\Pi) \cap \mathcal{C}_{B^+} = \text{CH}(\Pi)$. □

F Materials for Section C.4

F.1 Proof of Theorem 3

Theorem 3. (Semi-Random Likelihood of PMV-multi-instability, $B = O(n)$). *Given any $q \in \mathbb{N}$, any closed and strictly positive Π over $[q]$, and any union $\mathcal{M}$ of $I \in \mathbb{N}$ PMV-instability settings $\{\mathcal{S}^i : i \leq I\}$, there exists a constant $C_1 > 0$ so that for any $n \in \mathbb{N}$ and any $B \geq 0$ with $B \leq C_1 n$,*

$$\sup_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}^{\mathcal{M}}_{n,B}\right) = \begin{cases} 0 & \text{if } w_{\max} = -\infty \\ \exp(-\Theta(n)) & \text{if } w_{\max} = -\frac{2n}{\log n} \\ \Theta\left(\left(\frac{1}{\sqrt{n}}\right)^{q - w_{\max}}\right) & \text{otherwise} \end{cases}$$

$$\inf_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}^{\mathcal{M}}_{n,B}\right) = \begin{cases} 0 & \text{if } w_{\min} = -\infty \\ \exp(-\Theta(n)) & \text{if } w_{\min} = -\frac{2n}{\log n} \\ \Theta\left(\left(\frac{1}{\sqrt{n}}\right)^{q - w_{\min}}\right) & \text{otherwise} \end{cases}$$

Proof. The 0 cases of sup and inf are straightforward. Like the proof of Theorem 2, it suffices to prove the other cases of sup and inf for every sufficiently large n.

Proof for the sup part. We first prove a convenient corollary of Theorem 2, which uses the weight in the activation graph to represent conditions of the sup part of Theorem 2.

Corollary 2 (**Activation graph representation of the** sup **part of Theorem 2)**. *Given any $q \in \mathbb{N}$, any closed and strictly positive Π over $[q]$, and any PMV-instability setting $\mathcal{S} = \langle \mathcal{H}_S, \mathcal{H}_T, \mathbb{O}, \vec{c} \rangle$, any C_2 with $C_2 < B_{CH(\Pi)}$, any $n \in \mathbb{N}$, and any $0 \le B \le C_2 n$, let $w^* = \sup_{\pi \in CH(\Pi)} w_{n,B}(\pi, \mathcal{S})$,*

$$\sup_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}\right) = \begin{cases} 0 & \text{if } w^* = -\infty \\ \exp(-\Theta(n)) & \text{if } -\infty < w^* < 0 \\ \Theta\left(\left(\frac{1}{\sqrt{n}}\right)^{q-w^*}\right) & \text{otherwise} \end{cases}$$

Proof. The 0 case is straightforward. Like in the proof of Theorem 2, for the non-zero cases it is without loss of generality to assume that n is larger than a constant.

Exponential case. In this case we have $w^* = -\frac{2n}{\log n}$, which means that $\text{CH}(\Pi) \cap \mathcal{C}_0 = \emptyset$. Therefore, the exponential case of Corollary 2 follows after the exponential case and the $B \le C_2 n$ case of in Theorem 2.

Polynomial case. Because $\text{CH}(\Pi)$ is bounded and closed, it is compact. Therefore, there exists $\pi^* \in \text{CH}(\Pi)$ such that $w_{n,B}(\pi^*, \mathcal{S}) = w^* = d_0 + d_\Delta \cdot \min\left\{\frac{2\log(B+1)}{\log n}, 1\right\}$. It follows from Theorem 2 that

$$\sup_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}\right) = \Theta\left(\frac{\min\{B+1, \sqrt{n}\}^{d_\Delta}}{(\sqrt{n})^{q-d_0}}\right)$$

Notice that

$$\log\left(\frac{\min\{B+1, \sqrt{n}\}^{d_\Delta}}{(\sqrt{n})^{q-d_0}}\right) = d_\Delta \min\left\{\log(B+1), \frac{\log n}{2}\right\} + (d_0 - q)\frac{\log n}{2}$$
$$= \frac{\log n}{2} \cdot \left(d_0 + d_\Delta \cdot \min\left\{\frac{2\log(B+1)}{\log n}, 1\right\} - q\right) = \log\left(\left(\frac{1}{\sqrt{n}}\right)^{q-w^*}\right)$$

This completes the proof of Corollary 2. □

Define C_1 for sup. Let $C_1 > 0$ denote any positive number that is smaller than any strictly positive $B^i_{\text{CH}(\Pi)}$. That is,

$$0 < C_1 < \min\{B^i_{\text{CH}(\Pi)} : B^i_{\text{CH}(\Pi)} > 0, i \le I\}$$

The rest of the proof for the sup part of Theorem 3 is done by combining the results of the applications of Corollary 2 to all PMV-instability settings and the following inequality.

$$\max_{i \in I} \sup_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}^i_{n,B}\right) \le \sup_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}^{\mathcal{M}}_{n,B}\right) \le I \cdot \max_{i \in I} \sup_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}^i_{n,B}\right) \tag{26}$$

Exponential case of sup. In this case $w_{\max} = -\frac{2n}{\log n}$. Notice that for all $\pi \in \text{CH}(\Pi)$ and all $i \le I$, we have $w_{n,B}(\pi, \mathcal{S}^i) \le -\frac{2n}{\log n}$, and there exists $\pi^* \in \text{CH}(\Pi)$ and $i^* \le I$ such that $w_{n,B}(\pi^*, \mathcal{S}^{i^*}) = -\frac{2n}{\log n}$. Therefore, by Corollary 2, $\sup_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}^{i^*}_{n,B}\right) = -\frac{2n}{\log n}$, which means that

$$\max_{i \in I} \sup_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}^i_{n,B}\right) = \exp(-\Theta(n))$$

The exponential case follows after (26).

Polynomial case of sup. The proof for the polynomial case is similar. To prove the **polynomial upper bound**, notice that for all $\pi \in \text{CH}(\Pi)$ and all $i \le I$, we have $w_{n,B}(\pi, \mathcal{S}^i) \le w_{\max}$. By Corollary 2, when n is sufficiently large, for all $i \le I$, we have

$$\max_{i \in I} \sup_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}^i_{n,B}\right) = \max_{i \in I} \Theta\left(\left(\frac{1}{\sqrt{n}}\right)^{q-w^{i*}}\right) = O\left(\left(\frac{1}{\sqrt{n}}\right)^{q-w_{\max}}\right),$$

where w^{i*} is the weight w^* in Corollary 2 applied to $\mathcal{S} = \mathcal{S}^i$.

To prove the **polynomial lower bound**, let $\pi^* \in \text{CH}(\Pi)$ and $i^* \le I$ be such that $w^{i^*}_{n,B}(\pi^*, \mathcal{S}^{i^*}) = w_{\max}$. According to the polynomial case of Corollary 2, we have

$$\max_{i \in I} \sup_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}^i_{n,B}\right) = \Omega\left(\left(\frac{1}{\sqrt{n}}\right)^{q-w_{\max}}\right)$$

The polynomial lower bound of sup follows after (26). This proves the sup part of Theorem 3.

Proof for the inf part. The hardness in proving the inf part is that any $\vec{\pi} \in \Pi^n$ that achieves inf for one PMV-instability setting $\mathcal{S}^i$ may not achieve inf for another PMV-instability setting $\mathcal{S}^{i'}$, and therefore may not achieve inf of the PMV-multi-instability problem. This is different from the sup part, where the $\vec{\pi} \in \Pi^n$ with the highest value of sup under some $\mathcal{S}^{i'}$ achieves sup of the union-manipulation problem. Consequently, even though the inf counterpart of Corollary 2 can be proved for inf, it is cannot be leveraged to prove the inf part of Theorem 3.

To prove the inf part, we first define some distributions that will be used to prove the upper bounds for inf.

Definition 19 (**$\pi_{\mathcal{I}}$'s**). *Given a PMV-multi-instability setting $\mathcal{M}$, for every non-empty set $\mathcal{I} \subseteq [I]$ such that $CH(\Pi) \nsubseteq \bigcup_{i\in\mathcal{I}} C_0^i$, we choose $\pi_{\mathcal{I}} \in CH(\Pi) \setminus \left(\bigcup_{i\in\mathcal{I}} C_0^i\right)$.*

Because every C_0^i is a closed set, the distance between $\pi_{\mathcal{I}}$ and $\bigcup_{i\in\mathcal{I}} C_0^i$ is strictly positive and is denoted by $\delta_{\mathcal{I}} > 0$. Due to Claim 5, for every $i \in \mathcal{I}$ there exists a constant c_i such that each vector in $\mathcal{H}_B^i$ is no more than $c_i(B+1)$ away from a vector in C_0^i. It follows that for any $n > \frac{4c_i}{\delta_{\mathcal{I}}}$ and any $B \le \frac{\delta_{\mathcal{I}}}{4c_i} n$, the distance between $\pi_{\mathcal{I}}$ and $\mathcal{H}_B^i$ is at least $\frac{\delta_{\mathcal{I}}}{2} n$.

Define C_1 for inf. Let C_1 denote the minimum $\frac{\delta_{\mathcal{I}}}{4c_i}$ for all well-defined $\delta_{\mathcal{I}}$ and all $i \in \mathcal{I}$. Let δ denote the minimum $\frac{\delta_{\mathcal{I}}}{2}$ for all well-defined $\delta_{\mathcal{I}}$. It follows that for any sufficiently large n (that is larger than all $\frac{4c_i}{\delta_{\mathcal{I}}}$) and any $B \le C_1 n$, the distance between any well-defined $\pi_{\mathcal{I}}$ and any $i \in \mathcal{I}$ is at least δn.

Exponential case of inf. In this case $w_{\min} = -\frac{2n}{\log n}$. The **exponential lower bound** trivially holds, because there exists an active $\mathcal{S}^i$. To prove the **exponential upper bound**, let $\pi_{\text{MM}} \in \text{CH}(\Pi)$ be an arbitrary distribution such that for all $i \le I$, $w_{n,B}(\pi_{\text{MM}}, \mathcal{S}^i) \le -\frac{2n}{\log n}$. Let $\mathcal{I}_{\text{MM}}$ denote the indices to the active PMV-instability settings (whose C_0's do not contain π_{MM}), that is,

$$\mathcal{I}_{\text{MM}} \triangleq \left\{ i \le I : w_{n,B}(\pi_{\text{MM}}, \mathcal{S}^i) = -\frac{2n}{\log n} \right\}$$

Because $\pi_{\text{MM}} \in \text{CH}(\Pi) \setminus \left(\bigcup_{i\in\mathcal{I}} C_0^i\right)$, we have that $\pi_{\mathcal{I}_{\text{MM}}}$ is well-defined. Therefore, for every $i \in \mathcal{I}_{\text{MM}}$, the distance between $\pi_{\mathcal{I}_{\text{MM}}}$ and C_0^i is at least δn. The exponential upper bound follows after applying Claim 11, Hoeffding's inequality, and the union bound to any $\vec{\pi} \in \Pi^n$ such that $|\sum_{j=1}^n \pi_j - n \cdot \pi_{\mathcal{I}_{\text{MM}}}|_\infty = O(1)$, as done in the $B \le C_2 n$ case of the proof for PT-$\Theta(n)$-sup of Theorem 2.

Polynomial case of inf. To prove the **polynomial lower bound**, notice that for every $\vec{\pi} \in \Pi^n$, there exists $i \le I$ such that $w_{n,B}(\text{avg}(\vec{\pi}), \mathcal{S}^i) = d_{n,B}^i \ge w_{\min}$. It follows from Claim 10 (applied to $\mathcal{S}^i$, and $\vec{\pi}$) that, for every $B \le \sqrt{n}$,

$$\Pr(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}^{\mathcal{M}}) \ge \Pr(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}^i) = \Theta\left(\left(\frac{1}{\sqrt{n}}\right)^{q-d_{n,B}^i}\right) = \Omega\left(\left(\frac{1}{\sqrt{n}}\right)^{q-w_{\min}}\right)$$

Notice that for every $B > \sqrt{n}$, the inequality still holds, because we have $\Pr(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}^i) \ge \Pr(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,\sqrt{n}}^i)$.

To prove the **polynomial upper bound**, let

$$\pi_{\text{MM}} \triangleq \arg\min_{\pi\in\text{CH}(\Pi)} \max_{i\le I} \left\{ w_{n,B}(\pi, \mathcal{S}^i) \right\}$$

Like in the proof of the exponential upper bound of inf above, define

$$\mathcal{I}_{\text{MM}} \triangleq \left\{ i \le I : w_{n,B}(\pi_{\text{MM}}, \mathcal{S}^i) < 0 \right\}$$

Because $\pi_{\text{MM}} \in \text{CH}(\Pi) \setminus \left(\bigcup_{i\in\mathcal{I}} C_0^i\right)$, we have that $\pi_{\mathcal{I}_{\text{MM}}}$ is well-defined. Therefore, for every $i \in \mathcal{I}_{\text{MM}}$, the distance between $\pi_{\mathcal{I}_{\text{MM}}}$ and C_0^i is at least δn. Choose any $\vec{\pi} \in \Pi^n$ such that $|\sum_{j=1}^n \pi_j - n \cdot \pi_{\mathcal{I}_{\text{MM}}}|_\infty = O(1)$. Like the exponential case above, for every $i \in \mathcal{I}_{\text{MM}}$, we have $\Pr(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}^i) \le \exp(-\Omega(n))$. Moreover, for every $i \notin \mathcal{I}_{\text{MM}}$ such that $\mathcal{S}^i$ is active, we have $\pi_{\text{MM}} \in C_0^i$, which means that $C_0^i \cap \text{CH}(\Pi) \ne \emptyset$. Recall that $d_{n,B}^i \le w_{\min}$. By Corollary 2, we have

$$\Pr(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}^i) = O\left(\left(\frac{1}{\sqrt{n}}\right)^{q-d_{n,B}^i}\right) \le O\left(\left(\frac{1}{\sqrt{n}}\right)^{q-w_{\min}}\right)$$

Therefore,

$$\Pr(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}^{\mathcal{M}}) \le \sum_{i\le I} \Pr(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}^i) = O\left(\left(\frac{1}{\sqrt{n}}\right)^{q-w_{\min}}\right),$$

which proves the polynomial upper bound of inf. □

G Materials for Section 4

G.1 Full Version of Theorem 4 and Its Proof

THEOREM 4. *Let r be an integer positional scoring rule, STV, ranked pairs, Schulze, maximin, Copeland, plurality with runoff, or Bucklin with lexicographic tie-breaking. For any closed and strictly positive Π with $\pi_{uni} \in CH(\Pi)$, any $X \in \{\text{CM}, \text{MoV}\} \cup$ E-CONTROL (except $X \in$ E-CONTROL when $r = Cd_0$), there exists $N > 0$ such that for all $n > N$ and all $B \geq 0$,*

$$\widetilde{X}_{\Pi}^{\max}(r, n, B) = \begin{cases} 0 & \text{if } B = 0 \\ \Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}\right) & \text{if } B \geq 1 \end{cases}$$

For any $X \in$ CONTROL, any $n > N$ and any $B \geq 0$, $\widetilde{X}_{\Pi}^{\max}(r, n, B) = \Theta(1)$.

PROOF. We first prove the $\boldsymbol{B = 0}$ **case**. For any $X \in \{\text{CM}, \text{MoV}\} \cup$ E-CONTROL, it is not hard to verify that if no budget is given, then the goal under X cannot be reached, which requires the winner to be changed. Therefore, $\widetilde{X}_{\Pi}^{\max}(r, n, B) = 0$. For any $X \in$ CONTROL, it suffices to prove that for any alternative a,

$$\sup_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}}(r(P) = \{a\}) = \Theta(1) \tag{27}$$

It is not hard to see that for all voting rules mentioned in this statement of the theorem, there exists a polyhedron $\mathcal{H}^a$ such that for all $\vec{x} \in \mathcal{H}^a$, $r(\vec{x}) = \{a\}$, $\pi_{\text{uni}} \in \mathcal{H}^a_{\leq 0}$, and $\dim(\mathcal{H}^a_{\leq 0}) = m!$. Therefore, (27) follows after [77, Theorem 1] (or equivalently, Theorem 2 with $\mathcal{H}_{\text{S}} = \mathcal{H}_{\text{T}} = \mathcal{H}_a$ and $B = 0$).

In the rest of the proof, we assume that $X \in \{\text{CM}, \text{MoV}\} \cup$ E-CONTROL and $B \geq 1$.

Overview. Due to Theorem 5, it suffices to prove the $\Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}\right)$ matching lower bound by identifying a PMV-instability setting $\mathcal{S}$ that represents some unstable histograms $\mathcal{U}_{n,B}$, such that for all $B \geq 1$,

$$\widetilde{X}_{\Pi}^{\max}(r, n, B) \geq \sup_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}}(\text{Hist}(P) \in \mathcal{U}_{n,B}), \text{ and}$$

$$\sup_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}}(\text{Hist}(P) \in \mathcal{U}_{n,B}) = \Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}\right) \tag{28}$$

Notice that for any constant C_1 and any $B \geq C_1\sqrt{n}$, the right hands side of (28) is $\Theta(1)$. Therefore, it suffices to prove (28) for all $1 \leq B \leq C_1\sqrt{n}$. For each X and each voting rule r in the statement of the theorem, we define $\mathcal{S} = \langle \mathcal{H}_{\text{S}}, \mathcal{H}_{\text{T}}, \mathbb{O}, \vec{1} \rangle$, prove that $\mathcal{U}_{n,1} \neq \emptyset$ for any sufficiently large n by construction, and prove that $d_0 = m! - 1$ (which if often obvious), $d_\infty = m!$ (by applying Claim 15).

We now prove Theorem 4 for $X = \text{CM}$, and then comment on how to modify the proof for other $X \in \{\text{MoV}\} \cup$ E-CONTROL.

CM **for integer positional scoring rules.** Let $r = r_{\vec{s}}$ denote the positional scoring rule with scoring vector $\vec{s}$. Let $\mathcal{H}_{\text{S}}$ denote the set of vectors where 1's total score is at least as high as 2's total score, which is strictly higher than the total score of any other alternative. Let $\mathcal{H}_{\text{T}}$ denote the set of vectors where 2's total score is strictly the highest. To formally define $\mathcal{H}_{\text{S}}$ and $\mathcal{H}_{\text{T}}$, we first recall the definition of score difference vectors.

DEFINITION 20 (**Score difference vector [76]**). *For any scoring vector $\vec{s} = (s_1, \ldots, s_m)$ and any pair of different alternatives a, b, let $\text{Score}^{\vec{s}}_{a,b}$ denote the $m!$-dimensional vector indexed by rankings in $\mathcal{L}(\mathcal{A})$: for any $R \in \mathcal{L}(\mathcal{A})$, the R-element of $\text{Score}^{\vec{s}}_{a,b}$ is $s_{j_1} - s_{j_2}$, where j_1 and j_2 are the ranks of a and b in R, respectively.*

In words, $\text{Score}^{\vec{s}}_{a,b}$ is the score vector of a (under all linear orders) minus the score vector of b. Then, we define

$$\mathcal{H}_{\text{S}} \triangleq \left\{ \vec{x} : \begin{array}{c} \text{Score}^{\vec{s}}_{2,1} \cdot \vec{x} \leq 0 \\ \forall i \geq 3,\ \text{Score}^{\vec{s}}_{i,2} \cdot \vec{x} \leq -1 \\ -\vec{x} \leq \vec{0} \end{array} \right\}, \mathcal{H}_{\text{T}} \triangleq \left\{ \vec{x} : \begin{array}{c} \forall i \neq 2,\ \text{Score}^{\vec{s}}_{i,2} \cdot \vec{x} \leq -1 \\ -\vec{x} \leq \vec{0} \end{array} \right\}, \text{ and}$$

$$\mathcal{S}_{\vec{s}} = \langle \mathcal{H}_{\text{S}}, \mathcal{H}_{\text{T}}, \mathbb{O}^{1 \to 2}_{\pm}, \vec{1} \rangle$$

It is not hard to verify that for any $\vec{y} \in \mathcal{H}_{\text{S}}$ and any $\vec{x} \in \mathcal{H}_{\text{T}}$, we have $r_{\vec{s}}(\vec{y}) = \{1\}$ (1 has the highest score and wins due to tie-breaking if 2 also has the highest score) and $r_{\vec{s}}(\vec{x}) = \{2\}$ (2 has the strictly highest score).

Next, we show that $\mathcal{U}_{n,B} \neq \emptyset$ for any sufficiently large n by constructing a successful instance of manipulation by a single voter. We first define some profiles and rankings that will be used in the rest of the proof. For any $a \in \mathcal{A}$, let σ_a denote a cyclic permutation among $\mathcal{A} \setminus \{a\}$. Let P_a denote the following $(m-1)$-profile.

$$P_a \triangleq \left\{\sigma_a^i(a \succ \text{others}) : 1 \leq i \leq m-1\right\} \cup 3 \times \mathcal{L}(\mathcal{A}),$$

where alternatives in "others" are ranked alphabetically. Let

$$P_* \triangleq 2m \times (P_1 \cup P_2) \cup \bigcup_{i=3}^{m} 2(m-i) \times P_i$$

It follows that the $\vec{s}(P_*, 1) = \vec{s}(P_*, 2) > \vec{s}(P_*, 3) > \cdots > \vec{s}(P_*, m)$.

Let R_1 (respectively, R_2) denote the ranking where 1 (respectively, 2) is ranked at the top, 2 (respectively, 1) is ranked at the bottom, and the remaining alternatives are ranked alphabetically. That is,

$$R_1 \triangleq [1 \succ 3 \succ \cdots \succ m \succ 2] \text{ and } R_2 \triangleq [2 \succ 3 \succ \cdots \succ m \succ 1]$$

Let $\ell \le m-1$ denote the index to the minimum value of $s_\ell - s_{\ell+1}$. Let R_2' denote the ranking where 2 and 1 are ranked at the ℓ-th and the $(\ell+1)$-th positions respectively, and the remaining alternatives are ranked alphabetically. That is,

$$R_2' \triangleq \underbrace{3 \succ \cdots \succ \ell+1}_{\ell-1} \succ 2 \succ 1 \succ \underbrace{l+2 \succ \cdots \succ m}_{m-\ell-1}$$

Next, we define P_{S} and P_{T}. We first define P_{S}' to be the n-profile that consists of as many copies of P_* as possible, and the remaining rankings are R_1. That is, let $n' \triangleq \left\lfloor \frac{n}{|P_*|} \right\rfloor \times |P_*|$, and

$$P_{\mathrm{S}}' \triangleq \frac{n'}{|P_*|} \times P_* \cup (n - n') \times \{R_1\}$$

Let P_{S} denote the profile obtained from P_{S}' by replacing $\left\lfloor \frac{(n-n')(s_1-s_m)}{s_1+s_{\ell+1}-s_m-s_\ell} \right\rfloor$ copies of R_2' by R_2. It follows that for any sufficiently large n (so that P_{S}' contains enough copies of R_2' and the score difference between 1 and any alternative $i \ge 3$ is sufficiently large), P_{S} is well-defined and $r_{\vec{s}}(P_{\mathrm{S}}) = \{1\}$. Let P_{T} be obtained from P_{S} by replacing an R_2' vote by an R_2 vote. It follows that $r_{\vec{s}}(P_{\mathrm{T}}) = \{2\}$. This proves that $\mathcal{U}_{n,1} \neq \emptyset$ for any sufficiently large n.

It is not hard to verify that $d_0 = m! - 1$ (the only implicit equality represents 1 and 2 have the same score). Let P_{S}^* be the profile obtained from P_{S} by replacing an R_2 vote by an R_2' vote. It follows that $\text{Hist}(P_{\mathrm{S}}^*)$ and $\text{Hist}(P_{\mathrm{T}})$ are interior points of $\mathcal{H}_{\mathrm{S},\le 0}$ and $\mathcal{H}_{\mathrm{T},\le 0}$, respectively. Therefore, By Claim 15, $d_\infty = m!$. The lower bound (28) follows after the polynomial case of sup of Theorem 2 (applied to $\mathcal{S}_{\vec{s}}$). This completes the proof of Theorem 4 for CM under integer positional scoring rules.

CM **for STV.** Let $\mathcal{H}_{\mathrm{S}}$ consists of the histograms (for which the STV winner is 1) where the execution of STV satisfies the following conditions:

- for every $1 \le i \le m-4$, in round i, alternative $m+1-i$ has the strictly lowest plurality score among the remaining alternatives;
- In round $m-3$, alternative 3 has the highest score, and the score of 2 is no more than the score of 1;
- if 1 loses in round $m-3$, then 2 would become the winner; and if 2 loses in round $m-3$, then 1 would become the winner.

Formally, let us recall the score difference vector (for a pair of alternatives a, b, after a set of alternatives B is removed) to define $\mathcal{H}_{\mathrm{S}}$ and $\mathcal{H}_{\mathrm{T}}$.

Definition 21 ([78]). *For any pair of alternatives a, b and any subset of alternatives $B \subseteq (\mathcal{A} \setminus \{a, b\})$, we let $\text{Score}^\Delta_{B,a,b}$ denote the vector, where for every $R \in \mathcal{L}(\mathcal{A})$, the R-th component of $\text{Score}^\Delta_{B,a,b}$ is the plurality score of a minus the plurality score of b in R after alternatives in B are removed.*

Then, $\mathcal{H}_{\mathrm{S}}$ consists of vectors $\vec{x}$ that satisfies the following linear constraints.

- For every $i \le m-4$ and every $i' < i$,
$$\text{Score}^\Delta_{\{m+2-i,\ldots,m\},m+1-i,i'} \cdot \vec{x} \le -1$$
- Let $B_3 = \{4, \ldots, m\}$. There are two constraints: $\text{Score}^\Delta_{B_3,2,1} \cdot \vec{x} \le 0$ and $\text{Score}^\Delta_{B_3,1,3} \cdot \vec{x} \le -1$.
- $\text{Score}^\Delta_{B_3\cup\{1\},3,2} \cdot \vec{x} \le -1$ and $\text{Score}^\Delta_{B_3\cup\{2\},3,1} \cdot \vec{x} \le -1$.
- For every $R \in \mathcal{L}(\mathcal{A})$, there is a constraint $-x_R \le 0$.

Let $\mathcal{H}_{\mathrm{T}}$ denote the polyhedron that differs from $\mathcal{H}_{\mathrm{S}}$ in round $m-3$, where 1 has the lowest plurality score and drops out, which means that 2 is the STV winner. It is not hard to verify that for all $\vec{y} \in \mathcal{H}_{\mathrm{S}}$ and all $\vec{x} \in \mathcal{H}_{\mathrm{T}}$, we have $\text{STV}(\vec{y}) = \{1\}$ and $\text{STV}(\vec{x}) = \{2\}$.

Let $\mathcal{S}_{\text{STV}} \triangleq \langle \mathcal{H}_{\mathrm{S}}, \mathcal{H}_{\mathrm{T}}, \mathbb{O}_{\pm}^{1\to 2}, \vec{1} \rangle$. Next, we construct profiles P_{S} and P_{T} to show that $\mathcal{U}_{n,B} \neq \emptyset$ for any sufficiently large n and $B \ge 1$. For any $a \in \mathcal{A}$, let P_a^* denote the $(m-1)!$-profile that is obtained from $\mathcal{L}(\mathcal{A} \setminus \{a\})$ by putting a at the top. Let

$$P^* \triangleq \bigcup_{i=4}^{m} (m-i) \times P_a^* \text{ and } n^* \triangleq |P^*|$$

Let

$$P_{\mathrm{S}} \triangleq \left\lfloor \frac{n-n^*}{3} - 1 \right\rfloor \times \{[1 \succ 2 \succ \text{others}], [2 \succ 1 \succ \text{others}]\}$$
$$+ \left(n - n^* + 2 - 2\left\lfloor \frac{n-n^*}{3} \right\rfloor \right) \times \{3 \succ 2 \succ 1 \succ \text{others}\} + P^*$$

It follows that $|P_{\mathrm{S}}| = n$, and for all $n \ge n^* + 3m(m-1)!$, $\text{Hist}(P_{\mathrm{S}}) \in \mathcal{H}_{\mathrm{S}}$. Let P_{T} be the profile obtained from P_{S} by replacing one vote of $[3 \succ 2 \succ 1 \succ \text{others}]$ by $[2 \succ 3 \succ 1 \succ \text{others}]$. It is not hard to verify that $\text{Hist}(P_{\mathrm{T}}) \in \mathcal{H}_{\mathrm{T}}$. Therefore, $\mathcal{U}_{n,B} \neq \emptyset$ for every $B \ge 1$.

It is not hard to verify that $d_0 = \dim(\mathcal{H}_{\mathrm{S},\leq 0} \cap \mathcal{H}_{\mathrm{T},\leq 0}) = m! - 1$, where the only implicit equality requires 1 and 2 are tied for the last place in round $m-3$. To see $d_\infty = m!$, let P^*_{S} denote the profile obtained from P_{S} by adding one vote of $[1 \succ 2 \succ \text{others}]$. Let $\vec{y} = \mathrm{Hist}(P^*_{\mathrm{S}})$ and let $\vec{x}$ be the histogram of the profile obtained from P^*_{S} by changing two votes of $[3 \succ 2 \succ 1 \succ \text{others}]$ to $[2 \succ 1 \succ 3 \succ \text{others}]$. It is not hard to verify that $\vec{y}$ is an interior point of $\mathcal{H}_{\mathrm{S},\leq 0}$, $\vec{y}$ is an interior point of $\mathcal{H}_{\mathrm{T},\leq 0}$, $\dim(\mathcal{H}_{\mathrm{S},\leq 0}) = \dim(\mathcal{H}_{\mathrm{T},\leq 0}) = m!$. By Claim 15, we have $d_\infty = m!$.

Then, (28) follows after the application of the sup part of Theorem 2 to the PMV-instability setting $\mathcal{S}_{\mathrm{STV}}$ for CM under STV.

CM **for Ranked Pairs, Schulze, and maximin.** The proof for the three rules share the same construction. Let $\mathcal{H}_{\mathrm{S}}$ denote the polyhedron that consists of vectors $\vec{x}$ whose WMG satisfies the following conditions.

- The weights on the following edges are strictly positive: $1 \to 2$, $2 \to 3$, $3 \to 1$, $\{1,2,3\} \to \{4,\ldots,m\}$.
- For all $i \geq 4$, the weight on $1 \to i$ is strictly larger than the weight on $1 \to 2$.
- $w_{\vec{x}}(2 \to 3) > w_{\vec{x}}(1 \to 2) \geq w_{\vec{x}}(3 \to 1)$.

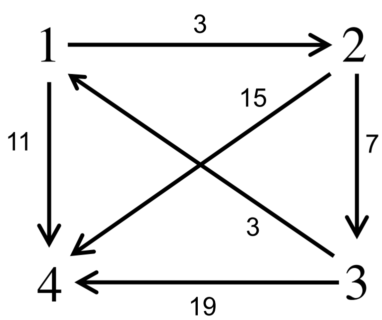
See Figure 5 (a) for an example of WMG that satisfies these conditions. Formally, we first recall the pairwise difference vectors as follows.

Definition 22 (**Pairwise difference vectors [76]**). *For any pair of different alternatives a, b, let $\mathrm{Pair}_{a,b}$ denote the $m!$-dimensional vector indexed by rankings in $\mathcal{L}(\mathcal{A})$: for any $R \in \mathcal{L}(\mathcal{A})$, the R-component of $\mathrm{Pair}_{a,b}$ is 1 if $a \succ_R b$; otherwise it is -1.*

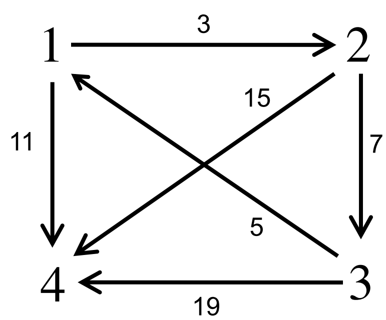
Then, let $\mathcal{H}_{\mathrm{S}}$ be characterized by the following linear inequalities/constraints:

- For each edge $a \to b \in \{1 \to 2, 2 \to 3, 3 \to 1\} \cup (\{1,2,3\} \to \{4,\ldots,m\})$, there is a constraint $\mathrm{Pair}_{b,a} \cdot \vec{x} \leq -1$.
- For all $i \geq 4$, $(\mathrm{Pair}_{1,2} - \mathrm{Pair}_{1,i}) \cdot \vec{x} \leq -1$.
- $(\mathrm{Pair}_{1,2} - \mathrm{Pair}_{2,3}) \cdot \vec{x} \leq -1$ and $(\mathrm{Pair}_{3,1} - \mathrm{Pair}_{1,2}) \cdot \vec{x} \leq 0$.
- For all linear order $R \in \mathcal{L}(\mathcal{A})$, there is a constraint $-x_R \leq 0$.

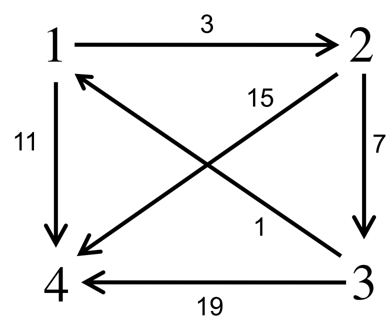
Let $\mathcal{H}_{\mathrm{T}}$ denote the polyhedron that consists of vectors $\vec{x}$ whose WMG satisfies the same conditions as $\mathcal{H}_{\mathrm{T}}$, except that now it is required that $w_{\vec{x}}(2 \to 3) > w_{\vec{x}}(3 \to 1) > w_{\vec{x}}(1 \to 2)$. See Figure 5 (b) for an example of WMG that satisfies these conditions for odd n. We have $\dim(\mathcal{H}_{\mathrm{T},\leq 0}) = m! - 1$ (the implicit equality is $(\mathrm{Pair}_{3,1} - \mathrm{Pair}_{2,3}) \cdot \vec{x} = 0$) and $\dim(\mathcal{H}_{\mathrm{T},\leq 0}) = m!$.



(a) $\mathrm{WMG}(P_{\mathrm{S}})$



(b) $\mathrm{WMG}(P_{\mathrm{T}})$



$\mathrm{WMG}(\vec{y})$

Figure 5: CM **under RP, Sch, and MM.**

Let $\mathcal{S} = \langle \mathcal{H}_{\mathrm{S}}, \mathcal{H}_{\mathrm{T}}, \mathbb{O}_{\pm}^{1\to 2}, \vec{1}\,\rangle$. It follows that for any $\vec{y} \in \mathcal{H}_{\mathrm{S}}$ and any $\vec{x} \in \mathcal{H}_{\mathrm{T}}$, $\mathrm{RP}(\vec{y}) = \{1\}$ ($1 \to 2$ is fixed before $3 \to 1$ due to tie-breaking) and $\mathrm{RP}(\vec{x}) = \{2\}$ ($3 \to 1$ is fixed before $1 \to 2$); $\mathrm{Sch}(\vec{y}) = \{1\}$ (1 and 2 are co-winners, so 1 wins due to tie-breaking) and $\mathrm{Sch}(\vec{x}) = \{2\}$ (2 is the unique winner); and $\mathrm{MM}(\vec{y}) = \{1\}$ (1 and 2 are co-winners, so 1 wins due to tie-breaking) and $\mathrm{MM}(\vec{x}) = \{2\}$ (2 is the unique winner).

Next, we define n-profiles P_{S} and P_{T} to show that $\mathcal{U}_{n,1} \neq \emptyset$. The construction depends on the parity of n. If n is odd, let P_{S} denote a profile whose WMG is the figure shown in Figure 5 (a), where the weights on $1 \to 2$ and $3 \to 1$ are the same. Due to McGarvey's theorem [44], such P_{S} exists for all sufficiently large odd number n, and we can assume that P_{S} contains two copies of $\mathcal{L}(\mathcal{A})$. Let P_{T} denote the profile obtained from P_{S} by replacing a $[2 \succ 1 \succ 3 \succ \text{others}]$ vote by $[2 \succ 3 \succ 1 \succ \text{others}]$, which means that the WMG of P_{T} is Figure 5 (b). If n is even, then let the positive weights on edges in the WMGs of P_{S} and P_{T} be one more than those for odd n, so that all weights become even numbers. In either case, it is not hard to verify that $\mathcal{U}_{n,B} \neq \emptyset$ for every sufficiently large n and every B.

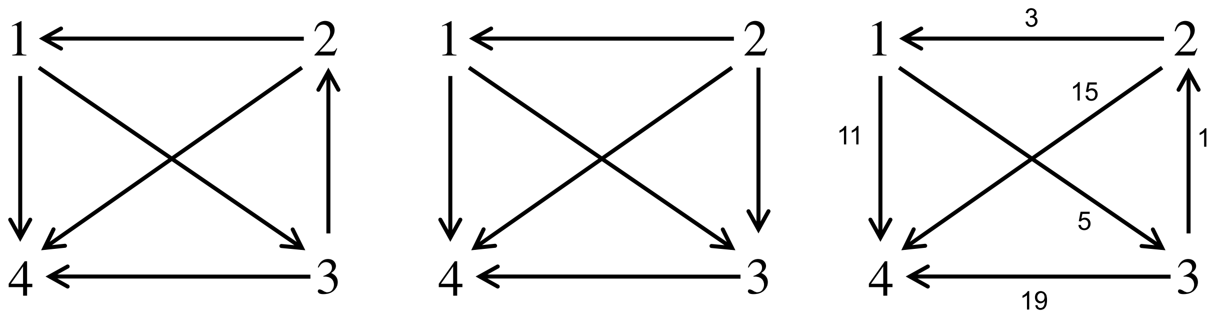
It is not hard to verify that $d_0 = m! - 1$. To see $d_\infty = m!$, let $\vec{y}$ be any vector such that $\mathrm{WMG}(\vec{x})$ is the same as Figure 5 (c). Let $\vec{x}$ denote the vector obtained from $\vec{y}$ by replacing two votes of $[2 \succ 1 \succ 3 \succ \text{others}]$ by $[2 \succ 3 \succ 1 \succ \text{others}]$. It follows that the $\mathrm{WMG}(\vec{x})$ is Figure 5 (b). Notice that $\vec{y}$ is an interior point of $\mathcal{H}_{\mathrm{S},\leq 0}$; $\vec{x}$ is an interior point of $\mathcal{H}_{\mathrm{T},\leq 0}$; and $\dim(\mathcal{H}_{\mathrm{S},\leq 0}) = \dim(\mathcal{H}_{\mathrm{T},\leq 0}) = m!$. By Claim 15, we have $d_\infty = m!$. Then, (28) follows after the application of the sup part of Theorem 2 to the PMV-instability setting for CM under ranked pairs, Schulze, and maximin.

CM **for Copeland.**

CM**, odd n.** Let $\mathcal{H}_S$ denote the polyhedron that consists of vectors $\vec{x}$ whose UMG has the following edges: $1 \to 3$, $3 \to 2$, $2 \to 1$, $\{1, 2, 3\} \to \{4, \ldots, m\}$. See Figure 6 (a) for an example for $m = 4$. Formally, $\mathcal{H}_S$ is characterized by the following linear inequalities/constraints:

- For each edge $a \to b \in \{1 \to 3, 3 \to 2, 2 \to 1\} \cup (\{1, 2, 3\} \to \{4, \ldots, m\})$, there is a constraint $\text{Pair}_{b,a} \cdot \vec{x} \leq -1$.
- For all linear order $R \in \mathcal{L}(\mathcal{A})$, there is a constraint $-x_R \leq 0$.

Let $\mathcal{H}_T$ denote the polyhedron that consists of vectors $\vec{x}$ whose UMG has the same edges $\mathcal{H}_S$, except that the direction between 2 and 3 is flipped, i.e., $w_{\vec{x}}(3 \to 2) \leq -1$. See Figure 6 (b) for an example of the UMG for $m = 4$.



(a) UMG for $\mathcal{H}_S$ (a) UMG for $\mathcal{H}_T$ (c) WMG(P_S)

Figure 6: CM **under** Cd_α**, odd** n**.**

Let $\mathcal{S}_{\text{Cd}_\alpha} = \langle \mathcal{H}_S, \mathcal{H}_T, \mathbb{O}_{\pm}^{1\to 2}, \vec{1} \rangle$. It follows that for any $\vec{y} \in \mathcal{H}_S$ and any $\vec{x} \in \mathcal{H}_T$, $\text{Cd}_\alpha(\vec{y}) = \{1\}$ (1, 2, 3 have the same highest Copeland score, and then 1 wins due to tie-breaking) and $\text{Cd}_\alpha(\vec{x}) = \{2\}$ (2 is the Condorcet winner).

Next, we define n-profiles P_S and P_T to show that $\mathcal{U}_{n,1} \neq \emptyset$. Let P_S denote a profile whose WMG is the figure shown in Figure 6 (c), where the weight on $3 \to 2$ is 1. Due to McGarvey's theorem [44], such P_S exists for all sufficiently large odd number n, and we can assume that P_S contains $\mathcal{L}(\mathcal{A})$. Let P_T denote the profile obtained from P_S by replacing a $[3 \succ 2 \succ 1 \succ \text{others}]$ vote by $[2 \succ 3 \succ 1 \succ \text{others}]$, which means that the UMG of P_T is Figure 6 (b).

Recall that P_T is obtained from P_S by replacing a vote. Therefore, $\text{Hist}(P_S) \in \mathcal{U}_{n,1}$, which means that $\mathcal{U}_{n,1} \neq \emptyset$. It is not hard to verify that $d_0 = m! - 1$ (the implicit equality corresponds to the tie between 2 and 3). Notice that $\text{Hist}(P_S)$ and $\text{Hist}(P_T)$ are interior points of $\mathcal{H}_{S,\leq 0}$ and $\mathcal{H}_{T,\leq 0}$, respectively, and $\dim(\mathcal{H}_{S,\leq 0}) = \dim(\mathcal{H}_{T,\leq 0}) = m!$. By Claim 15, we have $d_\infty = m!$. Also notice that $\pi_{\text{uni}} \in \mathcal{H}_{S,\leq 0} \cap \mathcal{H}_{T,\leq 0}$. Therefore, by Theorem 2, for any $B \geq 1$ and any sufficiently large odd n , we have

$$\sup_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}}(\text{Hist}(P) \in \mathcal{U}_{n,B}) = \Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}\right)$$

CM**, even n, $\alpha > 0$.** Let $\mathcal{H}_S$ be the same as defined for the odd n case above (the UMG of all vectors in $\mathcal{H}_S$ is illustrated in Figure 7 (a)). Let $\mathcal{H}_T$ be the polyhedron that consists of vectors $\vec{x}$ whose UMG satisfies the following conditions: the weights on the following edges are strictly positive: $1 \to 3$, $2 \to 1$, $\{1, 2, 3\} \to \{4, \ldots, m\}$. In addition, we require that $\text{WMG}(\vec{x})$ does not contain the edge $3 \to 2$, i.e., we require $w_{\vec{x}}(2 \to 3) \geq 0$. See Figure 7 (b) for the UMG for $m = 4$ (where the dashed edge from 2 to 3 means that either there is no edge between 2 and 3, or there is an edge $2 \to 3$). Formally, $\mathcal{H}_T$ is characterized by the following linear inequalities/constraints:

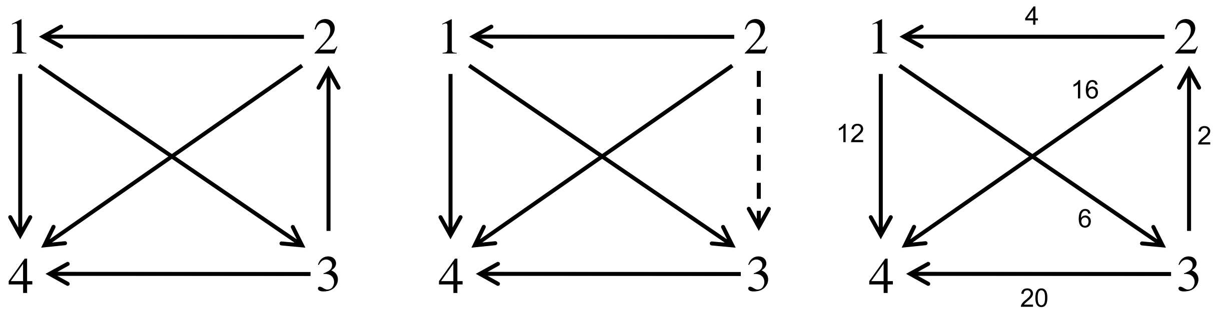
- For each edge $a \to b \in \{1 \to 3, 2 \to 1\} \cup (\{1, 2, 3\} \to \{4, \ldots, m\})$, there is a constraint $\text{Pair}_{b,a} \cdot \vec{x} \leq -1$.
- $\text{Pair}_{3,2} \cdot \vec{x} \leq 0$.
- For all linear order $R \in \mathcal{L}(\mathcal{A})$, there is a constraint $-x_R \leq 0$.

Let $\mathcal{S}_{\text{Cd}_\alpha} = \langle \mathcal{H}_S, \mathcal{H}_T, \mathbb{O}_{\pm}^{1\to 2}, \vec{1} \rangle$. It follows that for any $\vec{y} \in \mathcal{H}_S$ and any $\vec{x} \in \mathcal{H}_T$, $\text{Cd}_\alpha(\vec{y}) = \{1\}$ (1, 2, 3 have the same highest Copeland score, and then 1 wins due to tie-breaking) and $\text{Cd}_\alpha(\vec{x}) = \{2\}$ (2 has the highest Copeland score, which is at least $m - 2 + \alpha$).

Next, we define n-profiles P_S and P_T to show that $\mathcal{U}_{n,1} \neq \emptyset$. Let P_S denote any n-profile that contains $\mathcal{L}(\mathcal{A})$, whose UMG is as in Figure 7 (a), and its existence is guaranteed by McGarvey's theorem [44]. Let P_T denote the profile obtained from P_S by replacing a $[3 \succ 2 \succ 1 \succ \text{others}]$ vote by $[2 \succ 3 \succ 1 \succ \text{others}]$, which means that the UMG of P_T is consistent Figure 7 (b) (and there is no edge between 2 and 3 in $\text{UMG}(P_T)$).

To characterize d_∞, we prove the following convenient claim for general PMV-instability problems (for general q) that will be frequently used in the proofs of this paper.

Claim 15. *Suppose* $\dim(\mathcal{H}_{T,\leq 0}) = q$ *and* $\mathcal{H}_{S,\leq 0}$ *has an interior point that can be manipulated to an interior point of* $\mathcal{H}_{S,\leq 0}$*, then* $d_\infty = \dim(\mathcal{H}_{S,\leq 0})$.



(a) UMG for $\mathcal{H}_{\mathrm{S}}$ (b) UMG for $\mathcal{H}_{\mathrm{T}}$ (c) $\mathrm{WMG}(P_{\mathrm{S}})$

Figure 7: E-CCAV **under** $\mathbf{Cd}_\alpha$**, even** n**,** $\alpha > 0$**.**

Proof. Because $C_\infty \subseteq \mathcal{H}_{\mathrm{S},\leq 0}$, we have $d_\infty \leq \dim(\mathcal{H}_{\mathrm{S},\leq 0})$. Next, we prove that $d_\infty \geq \dim(\mathcal{H}_{\mathrm{S},\leq 0})$. Let $\vec{y} = \vec{x} - \vec{o} \times \mathbf{O}$ denote an interior point $\mathcal{H}_{\mathrm{S},\leq 0}$, where $\vec{x}$ is an interior point of $\mathcal{H}_{\mathrm{T},\leq 0}$ and $\vec{o} \geq \vec{0}$ represents successful operations (without budget constraints). Let $C > 0$ denote an arbitrary number so that the C neighborhood of $\vec{x}$ in L_∞ is contained in $\dim(\mathcal{H}_{\mathrm{T},\leq 0})$. Therefore, the C neighborhood of $\vec{y}$ in L_∞ is contained in $\mathcal{H}_{\mathrm{T},\leq 0} + Q_{\vec{c}\cdot\vec{o}}$. It follows that any vector $\vec{y}' \in \mathcal{H}_{\mathrm{S},\leq 0}$ that is at most C away from $\vec{y}$ in L_∞ is in C_∞. Because $\vec{y}$ is an interior point of $\mathcal{H}_{\mathrm{S},\leq 0}$, we have $d_\infty \geq \dim(\mathcal{H}_{\mathrm{S},\leq 0})$, which proves Claim 15. □

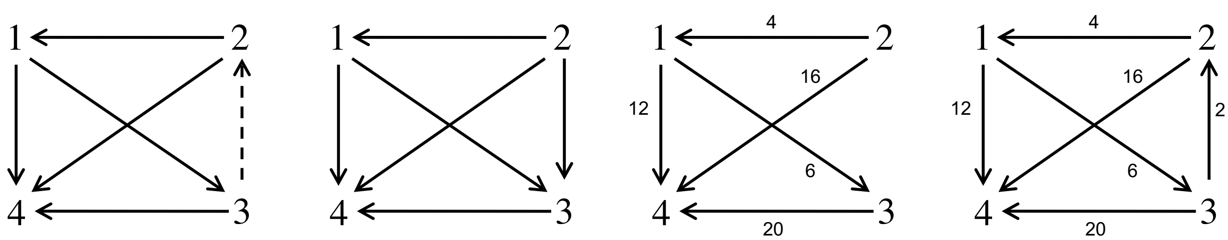
It is not hard to verify that $d_0 = m! - 1$ and $d_\infty = m!$ (by Claim 15). Notice that $\pi_{\mathrm{uni}} \in \mathcal{H}_{\mathrm{S},\leq 0} \cap \mathcal{H}_{\mathrm{T},\leq 0}$. The even n and $\alpha > 0$ case follows after Theorem 2.

CM**, even** $\boldsymbol{n}$**,** $\boldsymbol{\alpha = 0}$**.** Let $\mathcal{H}_{\mathrm{S}}$ be the polyhedron that consists of vectors $\vec{x}$ whose UMG contains $1 \to 3$, $2 \to 1$, $\{1,2,3\} \to \{4,\ldots,m\}$. In addition, we require that $\mathrm{WMG}(\vec{x})$ does not contain $2 \to 3$, that is, $w_{\vec{x}}(3 \to 2) \geq 0$. See Figure 8 (a) for the UMG for $m = 4$ (where the dashed edge from 3 to 2 means that either there is no edge between 2 and 3, or there is an edge $3 \to 2$). Formally, $\mathcal{H}_{\mathrm{S}}$ is characterized by the following linear inequalities/constraints:

- For each edge $a \to b \in \{1 \to 3, 2 \to 1\} \cup (\{1,2,3\} \to \{4,\ldots,m\})$, there is a constraint $\mathrm{Pair}_{b,a} \cdot \vec{x} \leq -1$.
- $\mathrm{Pair}_{2,3} \cdot \vec{x} \leq 0$.
- For all linear order $R \in \mathcal{L}(\mathcal{A})$, there is a constraint $-x_R \leq 0$.

Then, we let $\mathcal{H}_{\mathrm{T}}$ be the same as $\mathcal{H}_{\mathrm{T}}$ in the odd n case above (as illustrated in Figure 8 (b) for $m = 4$). Let $\mathcal{S}_{\mathrm{Cd}_\alpha} = \langle \mathcal{H}_{\mathrm{S}}, \mathcal{H}_{\mathrm{T}}, \mathbb{O}_{\pm}^{1\to 2}, \vec{1} \rangle$. It follows that for any $\vec{y} \in \mathcal{H}_{\mathrm{S}}$ and any $\vec{x} \in \mathcal{H}_{\mathrm{T}}$, $\mathrm{Cd}_\alpha(\vec{y}) = \{1\}$ (1 and 2 have the same highest Copeland score, so 1 wins due to tie-breaking) and $\mathrm{Cd}_\alpha(\vec{x}) = \{2\}$ (2 has the strictly highest Copeland score $m - 1$).

Let P_{S} denote an arbitrary n-profile whose WMG is as shown in Figure 8 (c) and it contains two copies of $\mathcal{L}(\mathcal{A})$. Let P_{T} denote the n-profile obtained from P_{S} by replacing a $[3 \succ 2 \succ 1 \succ \text{others}]$ vote by $[2 \succ 3 \succ 1 \succ \text{others}]$, which means that the UMG of P_{T} is like Figure 8 (b). It follows that $\mathrm{Hist}(P_{\mathrm{S}}) \in \mathcal{H}_{\mathrm{S}}$ and $\mathrm{Hist}(P_{\mathrm{T}}) \in \mathcal{H}_{\mathrm{T}}$, which proves that $\mathcal{U}_{n,1} \neq \emptyset$.



(a) UMG for $\mathcal{H}_{\mathrm{S}}$ (b) UMG for $\mathcal{H}_{\mathrm{T}}$ (c) $\mathrm{WMG}(P_{\mathrm{S}})$ (d) $\mathrm{WMG}(\vec{y})$

Figure 8: E-CCAV **under** $\mathbf{Cd}_\alpha$**, even** n**,** $\alpha = 0$**.**

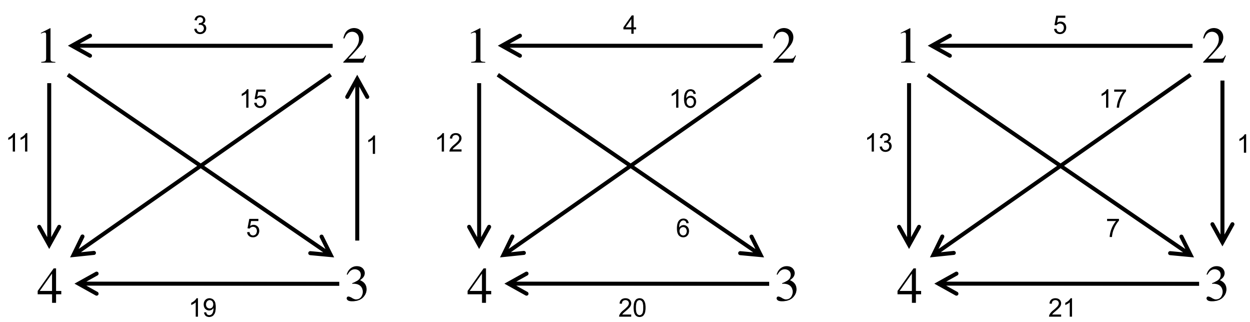
It is not hard to verify that $d_0 = m! - 1$. To see $d_\infty = m!$, let $\vec{y}$ be any vector such that $\mathrm{WMG}(\vec{x})$ is the same as Figure 8 (d) and it contains two copies of $[3 \succ 2 \succ 1 \succ \text{others}]$. Let $\vec{x}$ denote the vector obtained from $\vec{y}$ by replacing two votes of $[3 \succ 2 \succ 1 \succ \text{others}]$ vote by $[2 \succ 3 \succ 1 \succ \text{others}]$. It follows that the $\mathrm{UMG}(\vec{x})$ is Figure 8 (b). Notice that $\vec{y}$ is an interior point of $\mathcal{H}_{\mathrm{S},\leq 0}$; $\vec{x}$ is an interior point of $\mathcal{H}_{\mathrm{T},\leq 0}$; and $\dim(\mathcal{H}_{\mathrm{S},\leq 0}) = \dim(\mathcal{H}_{\mathrm{T},\leq 0}) = m!$. By Claim 15, we have $d_\infty = m!$. The even n and $\alpha = 0$ case follows after Theorem 2.

Other coalitional influence problems.

The proof for MoV is based on the same constructions of P_S, P_T, $\mathcal{H}_S$, and $\mathcal{H}_T$. The only difference is that the set of vote operations in $\mathcal{S}$ is $\mathbb{O}_\pm$.

The proofs for E-CCAV **and** E-CCDV are is based on similarly constructions of P_S, P_T, $\mathcal{H}_S$, and $\mathcal{H}_T$. The main differences are, first, the set of vote operations in $\mathcal{S}$ is $\mathbb{O}_+$ and $\mathbb{O}_-$ for E-CCAV and E-CCDV, respectively. Second, the added (respectively, deleted) votes correspond to the new (respectively, old) votes in CM. We add $3 \times \mathcal{L}(\mathcal{A})$ to P_S so that there is enough votes to be deleted for E-CCDV. Below we take constructive control $\{d\} = r(P_T)$ for example (where P_T depends on the problem and will be specified soon). Other cases can be proved similarly.

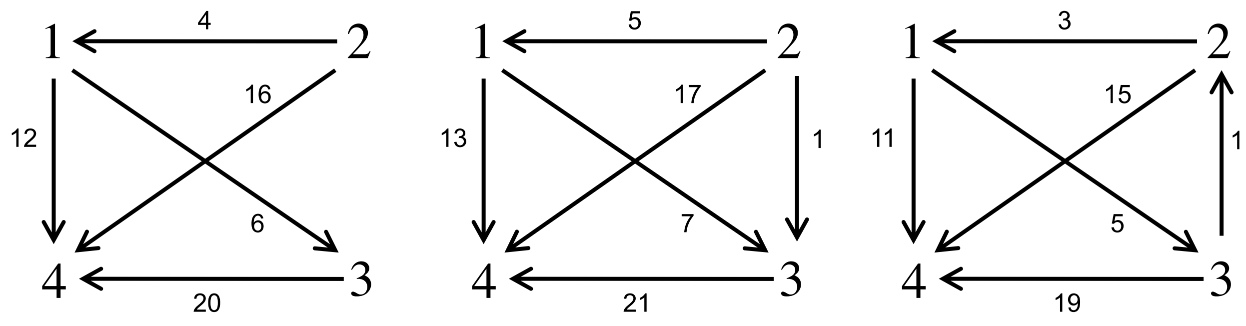
- **Integer positional scoring rules.** If $d \neq 1$, then P_S, $\mathcal{H}_S$, and $\mathcal{H}_T$ are similar to their counterparts in the proof of CM under integer positional scoring rules. Take $d = 2$ for example, for E-CCAV, the added votes are R_2; and for E-CCDV, the deleted votes are R_1. If $d = 1$, then we switch the roles of $\mathcal{H}_S$, and $\mathcal{H}_T$, and switch the roles of P_S and P_T in the proof of CM under integer positional scoring rules. Then, for E-CCAV, the added votes are R_1; and for E-CCDV, the deleted votes are R_2.
- **STV.** If $d \neq 1$, then P_S, $\mathcal{H}_S$, and $\mathcal{H}_T$ are similar to their counterparts in the proof of CM under STV. Take $d = 2$ for example, for E-CCAV, for E-CCAV, the added votes are $[2 \succ 3 \succ 1 \succ \text{others}]$; and for E-CCDV, the deleted votes are $[4 \succ 1 \succ 2 \succ 3 \succ \text{others}]$. If $d = 1$, then we switch the roles of $\mathcal{H}_S$, and $\mathcal{H}_T$, and switch the roles of P_S and P_T in the proof of CM under integer STV.
- **Ranked pairs, Schulze, and maximin.** If $d \neq 1$, then P_S, $\mathcal{H}_S$, and $\mathcal{H}_T$ are similar to their counterparts for CM. Take $d = 2$ for example, for E-CCAV, the added votes are $[2 \succ 3 \succ 1 \succ \text{others}]$; for E-CCDV, the deleted votes are $[1 \succ 3 \succ 2 \succ \text{others}]$. When n is even, the weights in WMG(P_S) are all even, for example the positive weights can be one plus the weights in Figure 5 (a). If $d = 1$, then we switch the roles of $\mathcal{H}_S$, and $\mathcal{H}_T$, and switch the roles of P_S and P_T in the proof of their counterparts for CM.
- **Copeland.** The proof for Copeland ($\alpha \neq 0$) is slightly more complicated than the proof for other rules, as $\mathcal{H}_S$, $\mathcal{H}_T$, P_S, and P_T depend on the parity of n. We prove E-CCAV with $d = 2$ (changed from a profile $\mathcal{H}_S$ where 1 is the winner) for odd and even n respectively, then comment on how to modify it for other cases.
  When *n* is odd, let $\mathcal{H}_S$ and $\mathcal{H}_T$ be the same as those for CM, even n, $\alpha > 0$ (Figure 7 (a) and (b)). Let P_S be any n-profile whose WMG is as shown in Figure 9 (a), and let P_T be the $(n+1)$-profile obtained from P_S by adding one vote of $[2 \succ 1 \succ 3 \succ \cdots \succ m]$. The WMG of P_T is shown in Figure 9 (b).



(a) WMG(P_S) (b) WMG(P_T) (c) WMG($\vec{x}$)

Figure 9: E-CCAV **under Cd_α, odd *n*.**

Let $\mathcal{S} = \langle \mathcal{H}_S, \mathcal{H}_T, \mathcal{S}_+, \vec{1} \rangle$. It follows that for any $\vec{y} \in \mathcal{H}_S$ and any $\vec{x} \in \mathcal{H}_T$, $\text{Cd}_\alpha(\vec{y}) = \{1\}$ (1, 2, and 3 have the same highest Copeland score, so 1 wins due to tie-breaking) and $\text{Cd}_\alpha(\vec{x}) = \{2\}$ (2 has the strictly highest Copeland score $m - 2 + \alpha$). It is not hard to verify that $d_0 = m! - 1$ (the only implicit equality is the tie between 2 and 3). Moreover, let $\vec{y} = \text{Hist}(\mathcal{H}_S)$ and let $\vec{x} = \text{Hist}(\mathcal{H}_S \cup 2 \times [2 \succ 1 \succ 3 \succ \cdots \succ m])$, whose WMG is illustrated in Figure 9 (c). Then we have that $\vec{y}$ and $\vec{x}$ are the interior points of $\mathcal{H}_{S,\leq 0}$ and $\mathcal{H}_{T,\leq 0}$, respectively, and it follows from Claim 15 that $d_\infty = m!$. The case of $d = 2$, odd n follows after Theorem 2.
When *n* is even, let $\mathcal{H}_S$ and $\mathcal{H}_T$ be the same as those for CM, even n, $\alpha = 0$ (Figure 8 (a) and (b)). Let P_S be any n-profile whose WMG is as shown in Figure 10 (a) (which is the same as 8 (c)), and let P_T be the $(n+1)$-profile obtained from P_S by adding one vote of $[2 \succ 1 \succ 3 \succ \cdots \succ m]$. The WMG of P_T is shown in Figure 10 (b) (which is the same as Figure 10 (c)).
Let $\mathcal{S} = \langle \mathcal{H}_S, \mathcal{H}_T, \mathcal{S}_+, \vec{1} \rangle$. It follows that for any $\vec{y} \in \mathcal{H}_S$ and any $\vec{x} \in \mathcal{H}_T$, $\text{Cd}_\alpha(\vec{y}) = \{1\}$ (1, 2, and 3 have the same highest Copeland score, so 1 wins due to tie-breaking) and $\text{Cd}_\alpha(\vec{x}) = \{2\}$ (2 has the strictly highest Copeland score $m - 2 + \alpha$). It is not hard to verify that $d_0 = m! - 1$ (the only implicit equality is the tie between 2 and 3). Moreover, let $\vec{y}$ denote the histogram of P_S subtracting one vote of $[2 \succ 1 \succ 3 \succ \cdots \succ m]$ (whose WMG is illustrated in Figure 10 (c), which is the same as Figure 9 (a)), and let $\vec{x} = \text{Hist}(\mathcal{H}_T)$. Then we have that $\vec{y}$ and $\vec{x}$ are the interior points of $\mathcal{H}_{S,\leq 0}$ and $\mathcal{H}_{T,\leq 0}$, respectively, and it follows from Claim 15 that $d_\infty = m!$. The case of $d = 2$, odd n follows after Theorem 2.



(a) WMG(P_{S}) (b) WMG(P_{T}) (c) WMG($\vec{y}$)

Figure 10: E-CCAV **under Cd_α, even n.**

The proof for E-CCAV **and any $d \geq 3$** is similar, which is done by simply switching the role 2 and d in the proof for $d = 2$. The proof for E-CCAV **and $d = 1$** is done by (1) switching the role of $\mathcal{H}_{\mathrm{S}}$ and $\mathcal{H}_{\mathrm{T}}$, (2) switching the role of P_{S} and P_{T}, and (3) the added vote is the inverse of $[2 \succ 1 \succ 3 \succ \cdots \succ m]$. The proof for E-CCDV is similar, by noticing that adding $[2 \succ 1 \succ 3 \succ \cdots \succ m]$ is equivalent to subtracting its inverse.

The proofs for E-DCAV **and** E-DCDV are similar to the proofs for E-CCAV and E-CCDV, as the proof essentially works for control of changing any source winner to any target winner. □

G.2 Full Version of Theorem 5 and Its Proof

THEOREM 5 (**Upper bound on Coalitional Influence under GSRs**). *Let r denote any GSR with fixed $m \geq 3$. For any closed and strictly positive Π, any $X \in \{\mathrm{CM}, \mathrm{MoV}\} \cup$ CONTROL $\cup$ E-CONTROL, any n, and any $B \geq 0$,*

$$\widetilde{X}^{\max}_{\Pi}(r, n, B) = O\left(\min\left\{\frac{B+1}{\sqrt{n}}, 1\right\}\right)$$

PROOF. Let X be any coalitional influence problem described in the lemma and let its PMV multi-instability representation be $\mathcal{M} = \{\mathcal{S}^i = \langle \mathcal{H}^i_{\mathrm{S}}, \mathcal{H}^i_{\mathrm{T}}, \mathbb{O}^i, \vec{c}^{\,i} \rangle : i \leq I\}$ due to Lemma 1. For every $i \leq I$, recall that there exist a pair of feasible signatures $\vec{t}_1$ and $\vec{t}_2$ so that $\overline{r}(\vec{t}_1) \neq \overline{r}(\vec{t}_2)$ and $\mathcal{H}^i_{\mathrm{S}} = \mathcal{H}_{\vec{t}_1}$ and $\mathcal{H}^i_{\mathrm{T}} = \mathcal{H}_{\vec{t}_2}$. Therefore, at least one component of $\vec{t}_1 \oplus \vec{t}_2$ is zero. This means that $d^i_0 = \dim(\mathcal{H}_{\vec{t}_1 \oplus \vec{t}_2}) \leq m! - 1$. Also notice that $d^i_\infty \leq m!$, which means that $d_\Delta \leq m! - d^i_0$. Therefore, according to Theorem 2, we have

$$\begin{aligned}&\sup_{\vec{\pi} \in \Pi^n} Pr_{P \sim \vec{\pi}}(\mathrm{Hist}(P) \in \mathcal{U}^i_{n,B}) = O\left(\min\{B+1, \sqrt{n}\}^{d^i_\Delta} \cdot \left(\frac{1}{\sqrt{n}}\right)^{m! - d^i_0}\right)\\ =&O\left(\left(\min\left\{\frac{B+1}{\sqrt{n}}, 1\right\}\right)^{m! - d^i_0}\right) = O\left(\min\left\{\frac{B+1}{\sqrt{n}}, 1\right\}\right)\end{aligned}$$

This proves Theorem 5 because I is finite. □

G.3 Proof of Theorem 6

THEOREM 6 (**Max-Semi-Random Coalitional Manipulation for The Loser**). *Let $r_{\vec{s}}$ be an integer positional scoring rule with lexicographic tie-breaking for fixed $m \geq 3$ that is different from veto. For any closed and strictly positive Π with $\pi_{uni} \in CH(\Pi)$, there exists $N > 0$ and $B^* > 0$ such that for any $n > N$ and any $B \geq B^*$,*

$$\widetilde{\mathrm{CML}}^{\max}_{\Pi}(r_{\vec{s}}, n, B) = \Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}^{m-1}\right)$$

PROOF. The proof proceeds in the following two steps.

Define the PMV-multi-instability setting $\mathcal{M}_{\mathrm{CML}}$. For every pair of different alternatives a, b, we define a PMV-instability setting $\mathcal{S}^{a \to b}_{\mathrm{CML}} = (\mathcal{H}_{\mathrm{S}}, \mathcal{H}_{\mathrm{T}}, \mathbb{O}^{a \to b}_{\pm}, \vec{1})$, where

- $\mathcal{H}_{\mathrm{S}}$ denote the set of vectors where a is the winner and b is the loser under $r_{\vec{s}}$.
- $\mathcal{H}_{\mathrm{T}}$ denote the set of vectors where b is the winner under $r_{\vec{s}}$.

Take $a = 1$ and $b = 2$ for example. Recall that $\text{Score}^{\vec{s}}_{a,b}$ is the score vector of b (under all linear orders) minus the score vector of a (Definition 20). Then, we have:

$$\mathcal{H}_{\text{S}} \triangleq \left\{ \vec{x} : \begin{array}{r} \forall i \geq 2,\ \text{Score}^{\vec{s}}_{i,1} \cdot \vec{x} \leq 0 \\ \forall i \geq 3,\ \text{Score}^{\vec{s}}_{2,i} \cdot \vec{x} \leq -1 \\ -\vec{x} \leq \vec{0} \end{array} \right\}, \mathcal{H}_{\text{T}} \triangleq \left\{ \vec{x} : \begin{array}{r} \text{Score}^{\vec{s}}_{1,2} \cdot \vec{x} \leq -1 \\ \forall i \geq 3,\ \text{Score}^{\vec{s}}_{i,2} \cdot \vec{x} \leq 0 \\ -\vec{x} \leq \vec{0} \end{array} \right\}, \text{ and}$$

$$\mathcal{M}_{\text{CML}} = \left\{ \mathcal{S}^{a\to b}_{\text{CML}} : a, b \in \mathcal{A}, a \neq b \right\}$$

Apply Theorem 2. In this step, we prove that for every $\mathcal{S}^{a\to b}_{\text{CML}}$ (with corresponding $\mathcal{U}^{a\to b}_{n,B}$),

$$\sup_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}\right) = \Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}^{m-1}\right) \tag{29}$$

We first prove $\kappa_1 = $ false (i.e., $\mathcal{U}^{a\to b}_{n,B} \neq \emptyset$) for any sufficiently large n and B, by constructing a successful manipulation by B voters. Recall from the proof of Theorem 4 that for any $a \in \mathcal{A}$, σ_a denotes a cyclic permutation among $\mathcal{A} \setminus \{a\}$. Then, we define two profiles, each of which consists of $m - 1$ votes as follows.

$$P^{\text{top}}_a \triangleq \left\{\sigma^i_a(a \succ \text{others}) : 1 \leq i \leq m-1\right\} \text{ and } P^{\text{bot}}_a \triangleq \left\{\sigma^i_a(\text{others} \succ a) : 1 \leq i \leq m-1\right\}$$

We further define the following "cyclic" profile of m votes: let σ denote any cyclic permutation among $\mathcal{A}$, e.g., $1 \to 2 \to \cdots \to m \to 1$.

$$P_{\text{cyc}} \triangleq \left\{\sigma^i(1 \succ \cdots \succ m) : 1 \leq i \leq m\right\}$$

Let P_{S} denote the n-profile that consists of P^{top}_a, P^{bot}_a, as many copies of P_{cyc} as possible, and the remaining rankings are $[a \succ \text{others} \succ b]$. Let P_{T} denote the profile obtained from P_{S} by replacing $\left\lceil \frac{(m-1)(s_1-s_m)}{(m-2)(s_1-s_{m-1})} \right\rceil + 1$ copies of $[\text{others} \succ b \succ a]$ to $[b \succ \text{others} \succ a]$. P_{T} is well-defined for any sufficiently large n. It is not hard to verify that a has the strictly highest score in P_{S}, b has the strictly lowest score in P_{S}, and b has the strictly highest score in P_{T}. This means that for any sufficient large B, $\mathcal{U}^{a\to b}_{n,B} \neq \emptyset$.

Next, let $\mathcal{C}^{a\to b}_0 = \mathcal{H}_{\text{S},\leq 0} \cap \mathcal{H}_{\text{T},\leq 0}$. We have $\pi_{\text{uni}} \in \text{CH}(\Pi) \cap \mathcal{C}_0$, which means that $\kappa_3 = $ false. Therefore, the polynomial case of Theorem 2 holds for $\mathcal{S}^{a\to b}_{\text{CML}}$. Notice that

$$\mathcal{C}^{a\to b}_0 = \mathcal{H}_{\text{S},\leq 0} \cap \mathcal{H}_{\text{T},\leq 0} = \left\{ \vec{x} : \begin{array}{r} \forall i \neq b,\ \text{Score}^{\vec{s}}_{i,b} \cdot \vec{x} \leq 0 \\ \forall i \neq b,\ \text{Score}^{\vec{s}}_{b,i} \cdot \vec{x} \leq 0 \\ -\vec{x} \leq \vec{0} \end{array} \right\}$$

Therefore, we have $d_0 = \dim(\mathcal{C}^{a\to b}_0) = m! - (m-1)$ (because the implicit equalities represent the scores of all alternatives are the same, which are characterized by $m - 1$ equations). Notice that $\text{Hist}(P_{\text{S}})$ is an interior point of $\mathcal{H}_{\text{S},\leq 0}$, $\text{Hist}(P_{\text{T}})$ is an interior point of $\mathcal{H}_{\text{T}}$, and $\dim(\mathcal{H}_{\text{S},\leq 0}) = \dim(\mathcal{H}_{\text{T},\leq 0}) = m!$. Therefore, by Claim 15, we have $d_\infty = \dim(\mathcal{H}_{\text{S},\leq 0}) = m!$, which means that $d_\Delta = m - 1$. It follows from Theorem 2 that

$$\sup_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}\right) = \Theta\left(\frac{\min\{B+1, \sqrt{n}\}^{m-1}}{(\sqrt{n})^{m!-(m!-(m-1))}}\right) = \Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}^{m-1}\right),$$

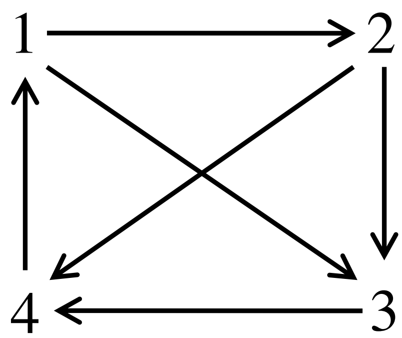
which proves Equation (29). Then, Theorem 6 follows after applying Equation (29) to all $a \neq b$. □

H Ties $\not\Leftrightarrow$ [$\Theta(1)$ instability]

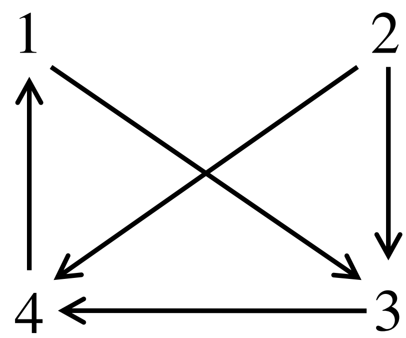
Example 6 (**Ties$\not\Rightarrow$ Θ(1) instability**). *Consider Cd_α with four alternatives. Let P' denote an arbitrary profile whose UMG is the same as Figure 11 (a). For any $n' \in \mathbb{N}$, we let $P = n'P'$. It is not hard to verify that $Cd_\alpha(P) = \{1, 2\}$. The winner under P is stable with $\Theta(n)$ changes in votes, because the UMG of any profile whose histogram is $\Theta(1)$ away from Hist(P) is the same as Figure 11 (a).*

Example 7 (**Ties$\not\Leftarrow$ Θ(1) instability**). *Let $\bar{r}$ denote a biased Copeland$_0$ rule for four alternatives, which differs from Copeland$_0$ in that if $1 \to 2$, then alternative 1 gets 2 points (instead of 1). Let P' denote an arbitrary profile whose UMG is the same as Figure 11 (b). For any $n' \in \mathbb{N}$, we let $P = n'P'$. Notice that $\bar{r}(P) = \{2\}$. To see that P is $\Theta(1)$ unstable, let R denote any vote in P where $2 \succ 1$. Replace R by $[1 \succ 2 \succ \text{others}]$, the winner becomes 1.*

Let P^* denote any profile that is $\Theta(1)$ away from P. It is not hard to see that UMG(P^*) contains the same edges as the graph in Figure 11 (b) except the edge between 1 and 2. This means that either $\bar{r}(P^*) = \{2\}$ (if there is no edge between 1 and 2 or there is an edge $2 \to 1$ in UMG(P^*)), or $\bar{r}(P^*) = \{1\}$ (if $1 \to 2$ in UMG(P^*)). This means that P is not close to any tied profile under $\bar{r}$.



(a) UMG for a tied stable profile.



(b) UMG for a un-tied unstable profile.

Figure 11: Graphs used in Example 6 and 7.